

Global model structures for \ast -modules

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ABSTRACT. This thesis gives a detailed account of model structures for unstable global homotopy theory. We introduce generalized fixed points functors and show how they give rise to a family of model structures for \mathcal{L} -spaces, including the sharp model structure which is monoidal and Quillen equivalent to the global model category of orthogonal spaces. We prove a theorem which transports model structures from \mathcal{L} -spaces to $*$ -modules and show that the resulting sharp model structure for $*$ -modules is monoidally Quillen equivalent to orthogonal spaces. There are induced Quillen equivalences between model structures on the associated categories of monoids and modules, which identify alternative models for the global homotopy theory of A_∞ -spaces.

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1. Introduction

Global homotopy theory is equivariant homotopy theory with respect to compatible actions of a family of groups rather than just a single group. We will concentrate on unstable global homotopy theory with respect to all compact Lie groups. It can be approached in several different ways, using \mathcal{L} -spaces, orthogonal spaces or orbispaces as a model.

Orthogonal spaces, sometimes called *\mathcal{I} -spaces*, are diagram spaces indexed on finite-dimensional inner product spaces. The equivariant structure comes from evaluation on group representations. Orthogonal spaces form a symmetric monoidal category \mathcal{IU} under the *box product* \boxtimes which is a special case of the Day convolution product, and they come with a notion of *global equivalence* which resembles equivalences detected on homotopy colimits. These equivalences are part of the *global model category* of orthogonal spaces which is compatible with the monoidal structure.

Our main model for global homotopy theory will be the category \mathcal{LU} of \mathcal{L} -spaces. These are spaces with an action of the monoid $\mathcal{L}(1)$ which is the space of unary operations in the linear isometries operad \mathcal{L} . The operadic structure gives rise to an operadic *box product* $\boxtimes_{\mathcal{L}}$ such that \mathcal{LU} is symmetric monoidal in a weaker sense than usual: The unitality condition does not hold for all objects. The category \mathcal{M}_* of **-modules* is the full subcategory of unital \mathcal{L} -spaces; it is an unstable analogue of the S -modules from [5], cf. Remark 5.12. All isomorphism classes of compact Lie groups have representatives which are genuine subgroups of $\mathcal{L}(1)$, so-called *universal subgroups*. These come in two flavours, depending on whether the action on \mathbb{R}^∞ is just faithful or even part of a complete universe. We will define \mathcal{C} -*equivalences* and \mathfrak{F} -*equivalences* of \mathcal{L} -spaces which are detected on sets of fixed points with respect to these two families of universal subgroups. In both cases, a global analogue of Elmendorf's Theorem [4, Thm. 1] holds and relates the associated \mathcal{C} -projective model category $(\mathcal{LU})_{\mathcal{C}}$ and the \mathfrak{F} -projective model category $(\mathcal{LU})_{\mathfrak{F}}$ to the categories $O_{\text{gl}}\mathcal{U}$ and $O_{\text{gl}}^{\mathfrak{F}}\mathcal{U}$, respectively, of *orbispaces*. The latter are two versions of diagram spaces indexed on a global orbit category.

Neither of these model structures on \mathcal{LU} is Quillen equivalent to \mathcal{IU} because there are not enough cofibrations, but these can be added in an intermediate step such that after a Bousfield localization, we obtain a *flat model structure* $(\mathcal{LU})_{\flat}$ with suitable cofibrations, and equivalences the \mathcal{C} -equivalences. It fits into a chain of Quillen equivalences

$$(\mathcal{U})_{\text{global}} \begin{array}{c} \xrightarrow{(-)(\mathbb{R}^\infty)} \\ \xleftarrow{u} \end{array} (\mathcal{U})_{\mathfrak{b}} \begin{array}{c} \xleftarrow{\text{id}} \\ \xrightarrow{\text{id}} \end{array} (\mathcal{U})_{\mathfrak{c}} \begin{array}{c} \xleftarrow{\Lambda} \\ \xrightarrow{\Phi} \end{array} (\mathcal{O}_{\text{gl}}\mathcal{U})_{\text{proj}}$$

which was established in [13, Sect. I.7]. The functor from orthogonal spaces to \mathcal{L} -spaces is strong symmetric monoidal, but unfortunately, $(\mathcal{U})_{\mathfrak{b}}$ is not a monoidal model category. On the other hand, $(\mathcal{U})_{\mathfrak{f}}$ is in fact monoidal, and so would be its flat replacement, but the \mathfrak{f} -equivalences are not compatible with the global equivalences of orthogonal spaces.

The two projective model structures are special cases of the \mathcal{C} -projective model structure, see Theorem A.30, on the category $M\mathcal{U}$ of spaces equipped with an action of the monoid M . It is parametrized by a family \mathcal{C} of well-behaved submonoids $N \leq M$ such that equivalences and fibrations are detected on N -fixed points. A set of generating cofibrations is given by $I_{\mathcal{C}} = \{M/N \times S^{k-1} \rightarrow M/N \times D^k \mid k \geq 0\}$. One could hope to use this fact to build a new, similar model structure on $\mathcal{L}\mathcal{U}$ by choosing the parameter \mathcal{C} such that

- (1) the weak equivalences are contained in the \mathfrak{c} -equivalences,
- (2) all cofibrations of orthogonal spaces become cofibrations in $\mathcal{L}\mathcal{U}$, and
- (3) the cofibrations are compatible with the box product, i.e., they are closed under forming pushout products.

It turns out that such a structure cannot be described in terms of fixed points anymore. Nevertheless, it exists and can be established in a way very similar to the \mathcal{C} -projective model structure, see Theorem A.34. This is what led us to the definition of a generalized fixed points functor in A.25, which shares all important properties of genuine fixed points and is still weak enough to encompass the kind of functors necessary to guarantee condition (3). We localize such that the equivalences become the \mathfrak{c} -equivalences again; the resulting *sharp model structure* $(\mathcal{L}\mathcal{U})_{\sharp}$ is symmetric monoidal and can replace the flat model structure in the above diagram. The adjunction on the left then becomes a monoidal Quillen equivalence.

We must not forget that $\mathcal{L}\mathcal{U}$ is a weak symmetric monoidal category; this inconvenience can be overcome by passage to $*$ -modules. So far, the homotopy theory of \mathcal{M}_* has only been studied non-equivariantly, with weak equivalences defined on underlying spaces, see [1, Sect. 4] and [8]. Following a hint in [13, Sect. I.7], we prove the Transport Theorem 5.3 which lifts well-behaved model structures on $\mathcal{L}\mathcal{U}$ to Quillen equivalent model structures on \mathcal{M}_* . It applies to the non-equivariant model structure as well as to all of the above model categories with weak equivalences the \mathfrak{c} - or

\mathfrak{F} -equivalences. In particular, there is a sharp model category $(\mathcal{M}_*)_{\sharp}$ of $*$ -modules which is symmetric monoidal. The functor from orthogonal spaces to \mathcal{L} -spaces factors over \mathcal{M}_* and forms a symmetric monoidal Quillen equivalence.

The Transport Theorem relies on the fact that all of our model categories are cofibrantly generated. This might fail for an arbitrary Bousfield localization because the fibrations may not be detected by a set of acyclic cofibrations anymore, but it is always true if the original model structure has the property of being *cellular*. We will make use of Hirschhorn's existence theorem [6, Thm. 4.1.1] for localizations of cellular model categories, which applies in our situation due to Proposition 4.11.

The categories of monoids and modules with respect to the box product $\boxtimes_{\mathcal{L}}$ of \mathcal{L} -spaces and the box product \boxtimes of orthogonal spaces admit model structures created by the forgetful functor to $(\mathcal{M}_*)_{\sharp}$ and $(\mathcal{U})_{\text{global}}$, respectively. These model categories are again Quillen equivalent via pairs of functors induced by the original adjunctions; the results are special cases of the main theorems of [14] and [15]. Since $\boxtimes_{\mathcal{L}}$ -monoids agree with A_{∞} -spaces structured by the linear isometries operad \mathcal{L} , the Quillen equivalences

$$\begin{array}{ccc}
 (\mathcal{U}\text{-monoids})_{\text{global}} & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & (\mathcal{L}\mathcal{U}\text{-monoids})_{\sharp} \\
 & \searrow \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & \begin{array}{c} \downarrow \\ \uparrow \end{array} \\
 & & (\mathcal{M}_*\text{-monoids})_{\sharp}
 \end{array}$$

imply that \boxtimes -monoids (called *orthogonal monoid spaces* in [13]) and monoids in \mathcal{M}_* provide alternative models for the homotopy theory of A_{∞} -spaces.

Organization:

Chapter 2 is a detailed exposition of the categories of \mathcal{L} -spaces and $*$ -modules and their (weak) symmetric monoidal behaviour. It introduces different notions of equivalences defined in terms of universal subgroups and the corresponding projective model structures. We will see some obstructions for a model category of \mathcal{L} -spaces to be compatible with the box product.

In *Chapter 3*, we briefly present the categories of orthogonal spaces and orbispaces, and their relation to \mathcal{L} -spaces. In particular, we work out what is necessary for a model structure on \mathcal{L} -spaces to provide the same homotopy theory as the other models.

We establish more sophisticated flat and sharp model categories of \mathcal{L} -spaces in *Chapter 4*. These stem from a Bousfield localization process and relate nicely to the other

models. Moreover, the sharp model structure is compatible with the (weak) monoidal structure.

Chapter 5 contains our main result, the “Transport Theorem”. It yields many model structures for $*$ -modules, one of which is a symmetric monoidal model category that is monoidally Quillen equivalent to orthogonal spaces. This structure lifts to model categories of monoids and modules, which are again Quillen equivalent to their analogues with respect to orthogonal spaces.

Appendix A provides a background on spaces with monoid actions and their homotopy theory. We introduce generalized fixed points functors and explain how these give rise to a collection of model structures.

Conventions:

In diagrams, the left adjoint of a pair of parallel functors will always be the upper one or the one on the left hand side, respectively. Diagrams are not assumed to commute in general.

We work over the category \mathcal{U} of compactly generated weak Hausdorff spaces and use its colimits and limits. In particular, \times will always denote the compactly generated product, subspaces carry the compactly generated subspace topology and mapping spaces carry the compactly generated compact-open topology. We will make frequent use of the fact that filtered colimits along closed embeddings can be computed in the category **Top** of *all* topological spaces, see [16, Lemma 3.3]. Maps are assumed to be continuous unless otherwise stated.

A *model category* is a (closed) Quillen model category as defined in [3, Def. 3.3]. The definition does not require functorial factorizations. Often, we will study several model structures on the same underlying category \mathcal{C} , which we denote as $(\mathcal{C})_a$, $(\mathcal{C})_b$, etc., where a, b, \dots are the names of the various model structures. When considering cofibrantly generated model categories, we use the language and definitions provided in [7, Sect. 2.1], the only difference being that from our point of view, the functorial factorizations given by the small object argument are not part of the data of the model category.

Most of the categories we work in are tensored over \mathcal{U} (as made precise in Proposition A.9) and we will usually describe sets of generating cofibrations and acyclic cofibrations in terms of the *standard generating cofibrations and acyclic cofibrations* for the Quillen model category of spaces. Recall that these are the sets of inclusions of subspaces

$$I^Q = \{i_k^Q: S^{k-1} \rightarrow D^k \mid k \geq 0\} \quad \text{and} \quad J^Q = \{j_k^Q: D^k \times \{0\} \rightarrow D^k \times I \mid k \geq 0\}$$

where S^{-1} is the empty set and I denotes the unit interval $[0, 1] \subseteq \mathbb{R}$. So we will often write $\{X \times i_k^{\mathcal{Q}}\}$ for a set of generating cofibrations, leaving implicit the (co-)domains of the morphisms and the condition $k \geq 0$.

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2. The category of \mathcal{L} -spaces

This section is devoted to our main model for global homotopy theory, the category of \mathcal{L} -spaces. An \mathcal{L} -space is a space equipped with an action by the monoid $\mathcal{L}(1)$ of unary operations of the operad of linear isometric embeddings. We will see that the isomorphism classes of compact Lie groups can be represented by *universal subgroups* of $\mathcal{L}(1)$. The operadic structure yields a *box product* of \mathcal{L} -spaces which is symmetric monoidal in a weak sense, and there is a natural choice of a subcategory of **-modules* which is then symmetric monoidal in the usual sense. It is a crucial part of our approach towards model structures for *-modules that this situation can be described in terms of monads. In the last section of this chapter, we define many kinds of equivalences by testing on fixed point spaces with respect to universal subgroups. They give rise to first examples of model structures for \mathcal{L} -spaces.

2.1. Basic definitions

We introduce a topological monoid $\mathcal{L}(1)$ of linear isometric self-embeddings of \mathbb{R}^∞ and define an \mathcal{L} -space to be a space with an $\mathcal{L}(1)$ -action. A general treatment of spaces with monoid actions can be found in Appendix A.

Let \mathcal{U} denote the category of compactly generated weak Hausdorff spaces. For convenience, we will simply refer to its objects as spaces. Recall that an *inner product space* is a real vector space together with an inner product, i.e., a non-degenerate, symmetric, positive-definite bilinear form. We will only consider inner product spaces of finite or countably infinite dimension.

Example 2.1 Let \mathbb{R}^∞ be the inner product space $\bigoplus_{\mathbb{N}} \mathbb{R}$ equipped with the standard inner product $\langle v, w \rangle = \sum_i v_i \cdot w_i$ for $v = (v_i)_{i \in \mathbb{N}}$, $w = (w_i)_{i \in \mathbb{N}}$.

Definition 2.2 Given two inner product spaces V and W , let $\mathbf{L}(V, W)$ denote the set of linear isometric embeddings from V to W .

We topologize $\mathbf{L}(V, W)$ as follows: For finite-dimensional V and W , it carries the subspace topology of $\mathcal{U}(V, W)$. If V is finite-dimensional and W of countably infinite dimension, we give $\mathbf{L}(V, W)$ the unique topology such that it becomes a colimit of the system $\mathbf{L}(V, W')$ for $W' \subseteq W$ finite-dimensional in the category \mathcal{U} . If V is countably infinite-dimensional, too, let $\mathbf{L}(V, W)$ be equipped with the topology such that it is an inverse limit of $\mathbf{L}(V', W)$ for $V' \subseteq V$ finite-dimensional.

Recall that an *operad* \mathcal{A} in a symmetric monoidal category is a collection of objects \mathcal{A}_n of n -ary operations for all $n \in \mathbb{N}$ together with composition maps which are subject to certain associativity and unitality laws. It is a *symmetric operad* if all \mathcal{A}_n come with free right Σ_n -actions from the symmetric group Σ_n such that certain equivariance conditions hold. See [10, Def. 1] for a precise definition.

Definition 2.3 Let \mathcal{L} be the operad of linear isometric embeddings. The space of n -ary operations $\mathcal{L}(n)$ is given by $\mathbf{L}((\mathbb{R}^\infty)^n, \mathbb{R}^\infty)$ and the operad composition is induced by the direct sum and the usual composition of linear maps. Using the action of Σ_n that permutes the n factors of $(\mathbb{R}^\infty)^n$, \mathcal{L} can be considered a symmetric operad.

Remark 2.4 All spaces $\mathcal{L}(n)$ are contractible, thus \mathcal{L} is an E_∞ -operad. Its algebras can be described in terms of the symmetric monoidal structure on \mathcal{L} -spaces defined in Section 2.3; see Section 5.5 for details.

Note that $\mathcal{L}(1)$ is a topological monoid under composition. We are now able to define \mathcal{L} -spaces.

Definition 2.5 An \mathcal{L} -space is a space $X \in \mathcal{U}$ together with a continuous $\mathcal{L}(1)$ -action. Write \mathcal{LU} for the category of \mathcal{L} -spaces and $\mathcal{L}(1)$ -equivariant maps.

The category \mathcal{LU} is a topological category, i.e., it is enriched over the category \mathcal{U} of spaces, see Section A.1. Due to formal reasons, it admits all colimits and limits, see Corollary A.13.

Example 2.6 The following examples will be important for us:

- 1) The spaces $\mathcal{L}(n)$ are \mathcal{L} -spaces via postcomposition. For $m, n \geq 1$, any choice of a linear isometric isomorphism $(\mathbb{R}^\infty)^m \cong (\mathbb{R}^\infty)^n$ determines an isomorphism $\mathcal{L}(m) \cong \mathcal{L}(n)$ of \mathcal{L} -spaces.
- 2) Every closed subgroup $G \leq \mathcal{L}(1)$ acts on $\mathcal{L}(1)$ from the right via precomposition. We write \mathcal{L}/G for the quotient \mathcal{L} -space $\mathcal{L}(1)/G := \mathcal{L}(1) \times_G *$, see Definition A.2.
- 3) We can consider $A \in \mathcal{U}$ as an \mathcal{L} -space equipped with the trivial $\mathcal{L}(1)$ -action.
- 4) We can associate to $A \in \mathcal{U}$ the *free \mathcal{L} -space* generated by A . It is given by $\mathcal{L}(1) \times A$ with $\mathcal{L}(1)$ acting trivially on the second factor (cf. Proposition A.12).

2.2. Universal subgroups

Let G be a compact Lie group. We will define G -universes in order to see that G is isomorphic to a so-called *universal subgroup* of the monoid $\mathcal{L}(1)$. Thus, \mathcal{L} -spaces are indeed equipped with compatible actions of all compact Lie groups.

We will always assume that group representations are orthogonal representations of finite or countably infinite dimension.

Definition 2.7 A G -universe U is an orthogonal G -representation of countably infinite dimension such that the following holds:

- i) It contains a trivial subrepresentation.
- ii) If a finite-dimensional G -representation V embeds into U , then $\bigoplus_{\mathbb{N}} V$ embeds into U as well.

The universe U is *faithful* if it is faithful as a G -representation. It is *complete* if every finite-dimensional G -representation embeds into it as a G -subrepresentation.

Clearly, each complete universe is a faithful universe.

Notation 2.8 Given a G -universe U , we write $s(U)$ for the poset of all finite-dimensional G -subrepresentations of U , ordered by inclusion.

A complete universe for G can be constructed as follows: Let $\text{Irr}(G)$ be the set of isomorphism classes of irreducible G -representations. The group G is compact, hence the cardinality of $\text{Irr}(G)$ is at most countable as a consequence of the theorem of Peter and Weyl, see [2, Exerc. III.4.7, 3]. Now

$$\bigoplus_{\mathbb{N}} \bigoplus_{[\rho] \in \text{Irr}(G)} \rho$$

is a complete G -universe. Each two complete G -universes are isomorphic as G -representations.

Example 2.9 Let U_G and U_K be universes for compact Lie groups G and K , respectively. Then $U = U_G \oplus U_K$ is a $(G \times K)$ -universe. If both U_G and U_K are faithful, so is U . Assume that both G and K are non-trivial, then U will never be complete, even if U_G and U_K are complete universes: Choose non-trivial irreducible representations ρ_G and ρ_K of G and K , respectively, then the $(G \times K)$ -representation $\rho = \rho_G \otimes \rho_K$ is irreducible and does not embed as a subrepresentation of U .

Definition 2.10 A compact subgroup $G \leq \mathcal{L}(1)$ is called *completely universal* (respectively *faithfully universal*) if it admits the structure of a compact Lie group (necessarily unique) such that under the tautological action, \mathbb{R}^∞ becomes a complete (respectively faithful) G -universe. For brevity, we will often speak of \mathfrak{C} -subgroups and \mathfrak{F} -subgroups.

Remark 2.11 Completely universal subgroups are discussed in [13, Sect. I.7], where they are just called universal subgroups.

Notation 2.12 Let $G \leq \mathcal{L}(1)$ be a \mathfrak{C} - or \mathfrak{F} -subgroup, then we write \mathbb{R}_G^∞ for the resulting universe under the tautological action on \mathbb{R}^∞ .

The following lemma gives a precise meaning of the slogan “ $\mathcal{L}(1)$ is the universal compact Lie group”, despite its being neither compact nor a group. It explains why \mathcal{L} -spaces indeed provide a model for spaces with compatible actions of all compact Lie groups.

Lemma 2.13 (cf. [13], I.6.) The conjugacy classes of \mathfrak{C} -subgroups of $\mathcal{L}(1)$ biject with the isomorphism classes of compact Lie groups.

Proof. Each compact Lie group G gives rise to a \mathfrak{C} -subgroup as follows: Choose a G -action on \mathbb{R}^∞ turning it into a complete G -universe. Then the image of the action homomorphism $G \rightarrow \mathbf{L}(\mathbb{R}_G^\infty, \mathbb{R}_G^\infty)$ is a \mathfrak{C} -subgroup isomorphic to G . As each two complete G -universes are related by a G -equivariant isometry $\phi \in \mathcal{L}(1)$, the resulting \mathfrak{C} -subgroups are conjugate via ϕ . All \mathfrak{C} -subgroups are in particular compact Lie groups; if they are conjugate in $\mathcal{L}(1)$, they are of course still isomorphic after forgetting the embedding into $\mathcal{L}(1)$. \square

Thus, \mathfrak{C} -subgroups should be thought of as nice representatives for isomorphism classes of compact Lie groups that sit inside $\mathcal{L}(1)$ and are unique up to conjugation.

Remark 2.14 In the same way, G can be replaced by any \mathfrak{F} -subgroup, but two such choices will not be conjugate in general. Still, the isomorphism classes of compact Lie groups biject with the *isomorphism* classes of \mathfrak{F} -subgroups.

2.3. The box product of \mathcal{L} -spaces

We define the *box product* $X \boxtimes_{\mathcal{L}} Y$ of \mathcal{L} -spaces and explain that the category \mathcal{LU} is symmetric monoidal in a slightly weaker sense than what is usually called a symmetric

monoidal category. Our exposition is based on [1, Ch. 4], but differs from it in some of the details.

Definition 2.15 The *box product* (or *operadic product*) of \mathcal{L} -spaces X and Y is the choice of a balanced product (see Definition A.2)

$$X \boxtimes_{\mathcal{L}} Y := \mathcal{L}(2) \times_{\mathcal{L}(1)^2} (X \times Y)$$

with $\mathcal{L}(1)$ -action coming from postcomposition on $\mathcal{L}(2)$.

Example 2.16 ([5], Lemma I.5.4) For $m, n \geq 1$, the diagram

$$\mathcal{L}(2) \times \mathcal{L}(1)^2 \times (\mathcal{L}(m) \times \mathcal{L}(n)) \rightrightarrows \mathcal{L}(2) \times (\mathcal{L}(m) \times \mathcal{L}(n)) \longrightarrow \mathcal{L}(m+n)$$

is a split coequalizer of spaces, hence $\mathcal{L}(m) \boxtimes_{\mathcal{L}} \mathcal{L}(n) \cong \mathcal{L}(m+n)$ as \mathcal{L} -spaces. According to [5], this observation is due to Hopkins.

Example 2.17 Let $G, K \leq \mathcal{L}(1)$ be \mathfrak{F} -subgroups. We compute the box product of \mathcal{L}/G with \mathcal{L}/K . There is an isomorphism of \mathcal{L} -spaces

$$\mathcal{L}/G \boxtimes_{\mathcal{L}} \mathcal{L}/K = \mathcal{L}(2) \times_{\mathcal{L}(1)^2} (\mathcal{L}/G \times \mathcal{L}/K) \cong \mathbf{L}(\mathbb{R}_G^\infty \oplus \mathbb{R}_K^\infty, \mathbb{R}^\infty) / (G \times K)$$

where $G \times K$ acts on $\mathbf{L}(\mathbb{R}_G^\infty \oplus \mathbb{R}_K^\infty, \mathbb{R}^\infty)$ in the obvious way: $(\psi_1 \oplus \psi_2)(g, k) = (\psi_1 g, \psi_2 k)$. The vector space $\mathbb{R}_G^\infty \oplus \mathbb{R}_K^\infty$ is a faithful $(G \times K)$ -universe and we may choose an equivalent $(G \times K)$ -action on \mathbb{R}^∞ to obtain another faithful universe $\mathbb{R}_{G \times K}^\infty$. Let H be the image of the action map $G \times K \rightarrow \mathbf{L}(\mathbb{R}_{G \times K}^\infty, \mathbb{R}_{G \times K}^\infty)$, then $H \leq \mathcal{L}(1)$ is an \mathfrak{F} -subgroup which is isomorphic to $G \times K$. Any choice of an equivariant isometric isomorphism $\mathbb{R}_G^\infty \oplus \mathbb{R}_K^\infty \cong \mathbb{R}_{G \times K}^\infty$ induces an isomorphism of \mathcal{L} -spaces

$$\mathbf{L}(\mathbb{R}_G^\infty \oplus \mathbb{R}_K^\infty, \mathbb{R}^\infty) / (G \times K) \cong \mathbf{L}(\mathbb{R}_{G \times K}^\infty, \mathbb{R}^\infty) / H = \mathcal{L}/H,$$

hence up to isomorphism, the class $\{\mathcal{L}/G\}$ for \mathfrak{F} -subgroups G is closed under forming box products.

Example 2.18 The class of \mathfrak{C} -subgroups is not closed under box product: $\mathbb{R}_{K_1}^\infty \oplus \mathbb{R}_{K_2}^\infty$ is a faithful universe for $K_1 \times K_2$, but it is not complete (except for the degenerate cases $K_1 = \{\text{id}\}$ or $K_2 = \{\text{id}\}$). We can replace $K_1 \times K_2$ by an isomorphic \mathfrak{F} -subgroup K such that $\mathcal{L}/K_1 \boxtimes_{\mathcal{L}} \mathcal{L}/K_2 \cong \mathcal{L}/K$. If G is a \mathfrak{C} -subgroup replacement for K , then the spaces \mathcal{L}/G and \mathcal{L}/K are weakly equivalent, see Corollary 2.54, but it seems unlikely that they will ever be isomorphic as \mathcal{L} -spaces, which we prove for *finite* groups:

Assume, there is an isomorphism $\Phi: \mathcal{L}/G \rightarrow \mathcal{L}/K$ and write $\Phi(\text{id} \cdot G) = \theta \cdot K$. The monoid $\mathcal{L}(1)$ acts transitively on $\text{id} \cdot K$ and $\theta \cdot K$, hence the stabilizers of these two orbits are conjugate via the invertible element $\theta \in \mathcal{L}(1)$. Lemma A.5 implies that these stabilizers are just the groups K and G , respectively. The class of \mathfrak{C} -subgroups is closed under conjugation, hence K must be a \mathfrak{C} -subgroup, which is absurd.

For any $Y \in \mathcal{LU}$, the functor $- \boxtimes_{\mathcal{L}} Y: \mathcal{LU} \rightarrow \mathcal{LU}$ has a right adjoint internal mapping space functor: Note that there is a right action

$$\mathcal{L}(1)^2 \times \mathcal{L}(2) \rightarrow \mathcal{L}(2), \quad (\phi_1, \phi_2, \psi) \mapsto \psi \circ (\phi_1 \oplus \phi_2)$$

on $\mathcal{L}(2)$ via precomposition. Restriction along the inclusions $\mathcal{L}(1) \times \{\text{id}\} \rightarrow \mathcal{L}(1)^2$ (or $\{\text{id}\} \times \mathcal{L}(1) \rightarrow \mathcal{L}(1)^2$, respectively) yields two right $\mathcal{L}(1)$ -actions which only alter a map on the first (respectively second) summand of $\mathbb{R}^\infty \oplus \mathbb{R}^\infty$. We will refer to these actions as the first (respectively second) action on $\mathcal{L}(2)$; they induce a first and second left action on $\mathcal{LU}(\mathcal{L}(2), Y)$. We are now ready to define the right adjoint functor.

Definition 2.19 Let $F_{\boxtimes_{\mathcal{L}}}(Y, Z)$ be the \mathcal{L} -space with underlying space

$$F_{\boxtimes_{\mathcal{L}}}(Y, Z) = \mathcal{LU}(Y, \mathcal{LU}(\mathcal{L}(2), Z)).$$

Here, $\mathcal{LU}(\mathcal{L}(2), Z)$ is the space of $\mathcal{L}(1)$ -equivariant maps with respect to the left $\mathcal{L}(1)$ -actions on $\mathcal{L}(2)$ and Z . It is an \mathcal{L} -space with respect to the second action coming from $\mathcal{L}(2)$. Finally, the $\mathcal{L}(1)$ -action on $F_{\boxtimes_{\mathcal{L}}}(Y, Z)$ is induced by the first action.

Remark 2.20 For $Y = *$, we can identify the underlying space of $F_{\boxtimes_{\mathcal{L}}}(*, Z)$ with the $\mathcal{L}(1)$ -fixed points $\mathcal{LU}(\mathcal{L}(2), Z)^{\mathcal{L}(1)}$ taken with respect to the second action induced by the right $\mathcal{L}(1)^2$ -space $\mathcal{L}(2)$.

Lemma 2.21 The functor $F_{\boxtimes_{\mathcal{L}}}(Y, -): \mathcal{LU} \rightarrow \mathcal{LU}$ is a right adjoint for the functor $- \boxtimes_{\mathcal{L}} Y: \mathcal{LU} \rightarrow \mathcal{LU}$.

Proof. Compose the two adjunctions A.14 and A.15 and choose the monoids as follows: $M = M' = M'' = \mathcal{L}(1)$, $N = \mathcal{L}(1)^2$. □

We will take a closer look at $\mathcal{L}(2)$ in order to define some natural transformations related to the box product. The space $\mathcal{L}(2)$ embeds $\mathcal{L}(1)$ -equivariantly into the space of all linear maps $\text{Lin}((\mathbb{R}^\infty)^2, \mathbb{R}^\infty)$. By the universal property of the direct sum,

$\text{Lin}((\mathbb{R}^\infty)^2, \mathbb{R}^\infty) \cong \text{Lin}(\mathbb{R}^\infty, \mathbb{R}^\infty)^2$, and under this identification, the embedding of $\mathcal{L}(2)$ takes values in $\mathcal{L}(1)^2 \subseteq \text{Lin}(\mathbb{R}^\infty, \mathbb{R}^\infty)^2$. Hence, there is an $\mathcal{L}(1)$ -equivariant map $\beta: \mathcal{L}(2) \rightarrow \mathcal{L}(1)^2$ with respect to the diagonal action on the target. We will write $(\psi_1, \psi_2) := \beta(\psi)$ for the image of $\psi \in \mathcal{L}(2)$. In other words, ψ_i is the restriction of ψ along the inclusion $\mathbb{R}^\infty \rightarrow (\mathbb{R}^\infty)^2$ of the i -th summand.

An easy calculation shows that the composite

$$\begin{array}{ccc} \mathcal{L}(2) \times (X \times Y) & \xrightarrow{\beta \times \text{id}} & \mathcal{L}(1) \times \mathcal{L}(1) \times (X \times Y) \\ & & \downarrow \text{swap} \\ & & (\mathcal{L}(1) \times X) \times (\mathcal{L}(1) \times Y) \xrightarrow{\text{act} \times \text{act}} X \times Y \end{array}$$

descends to a natural morphism of \mathcal{L} -spaces

$$\lambda_{X,Y}: X \boxtimes_{\mathcal{L}} Y \rightarrow X \times Y, [\psi, (x, y)] \mapsto (\psi_1 \cdot x, \psi_2 \cdot y). \quad (2.22)$$

Write λ_X for the natural map

$$X \boxtimes_{\mathcal{L}} * \xrightarrow{\lambda_{X,*}} X \times * \xrightarrow{\cong} X \quad (2.23)$$

Definition 2.24 Let $\bar{\lambda}_X: X \rightarrow F_{\boxtimes_{\mathcal{L}}}(*, X)$ be the natural map corresponding to the map $\lambda_X: X \boxtimes_{\mathcal{L}} * \rightarrow X$ under the adjunction isomorphism

$$\mathcal{LU}(X \boxtimes_{\mathcal{L}} *, X) \cong \mathcal{LU}(X, F_{\boxtimes_{\mathcal{L}}}(*, X)).$$

Lemma 2.25 Explicitly, $\bar{\lambda}_X$ sends $x \in X$ to the map corresponding to the $\mathcal{L}(1)$ -fixed point $(\psi \mapsto \psi_1 \cdot x) \in \mathcal{LU}(\mathcal{L}(2), X)^{\mathcal{L}(1)} \cong F_{\boxtimes_{\mathcal{L}}}(*, X)$.

Analogously to [5, Thm. I.8.5], Blumberg, Cohen and Schlichtkrull discovered that λ_X is always a weak homotopy equivalence of underlying spaces see [1, Prop. 4.5]. In fact, the same is true for all maps $\lambda_{X,Y}$, and both natural maps satisfy a much stronger notion of equivalence, as we will see in Section 2.6. Similar statements for $\bar{\lambda}_X$ can be derived formally.

Without having seen model structures for \mathcal{L} -spaces so far, the reader may think of \mathcal{LU} as a *relative category* for the moment, i.e., a category together with a distinguished class of weak equivalences \mathcal{W} which is closed under composition and retracts, contains all isomorphisms and satisfies the 2-out-of-3 property. For now, \mathcal{W} is just the class of weak homotopy equivalences. The benefits and deficits of different choices for \mathcal{W} will become apparent in the course of our exposition of model structures. We can weaken the definition of a symmetric monoidal category to give an adequate description of \mathcal{LU} under the box product.

Definition 2.26 Let $(\mathcal{C}, \mathcal{W})$ be a relative category as above, equipped with a biproduct $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and a unit object $\mathbb{1} \in \mathcal{C}$, then it is a *weak symmetric monoidal category* if it satisfies all axioms of a symmetric monoidal category except that the unital transformation $X \otimes \mathbb{1} \rightarrow X$ (and thus $\mathbb{1} \otimes X \rightarrow X$) may not be an isomorphism in general, but only an element of \mathcal{W} . It is *closed* if for each $X \in \mathcal{C}$, the functor $- \otimes X$ has a right adjoint.

A weak symmetric monoidal category with respect to the class \mathcal{W} of all isomorphisms is just a symmetric monoidal category in the usual sense.

Proposition 2.27 The category \mathcal{LU} together with the class of weak homotopy equivalences (of underlying maps of spaces) is a closed weak symmetric monoidal category with respect to the box product $\boxtimes_{\mathcal{L}}$ and the monoidal unit $* \in \mathcal{LU}$.

This is proven in [1, Sect. 4.1], mimicking many ideas from [5, Sect. I.5]. The statement is still true if the class of weak equivalences is replaced by the class of \mathfrak{F} -equivalences or any of the weaker notions of an equivalence of \mathcal{L} -space, see Lemma 2.51. However, the map λ_X is certainly not an isomorphism for all \mathcal{L} -spaces X as shown in Example 2.32.

Remark 2.28 We warn the reader that the unital transformation sometimes appears as a map $* \boxtimes_{\mathcal{L}} X \rightarrow X$ in the literature. From our perspective, this map is obtained by precomposing λ_X with the commutativity isomorphism $\tau: * \boxtimes_{\mathcal{L}} X \rightarrow X \boxtimes_{\mathcal{L}} *$. In view of the adjunction isomorphism $\mathcal{LU}(X \boxtimes_{\mathcal{L}} *, X) \cong \mathcal{LU}(X, F_{\boxtimes_{\mathcal{L}}}(*, X))$, it seems a more natural choice to write down the singleton on the right hand side of the product.

2.4. The category of $*$ -modules

The inconvenience of working with a *weak* monoidal category can be avoided easily.

Definition 2.29 Let $\mathcal{M}_* \subset \mathcal{LU}$ be the full subcategory of *unital* \mathcal{L} -spaces, i.e., those \mathcal{L} -spaces X such that λ_X is an isomorphism. Its objects are called $*$ -modules.

The category of $*$ -modules is the unstable analogue of S -modules. In fact, the definition of the first mimicks the definition of the latter, cf. [5, Ch. 2], but starts with \mathcal{L} -objects in \mathcal{U} rather than in a category of spectra. The stable and unstable worlds are related by an infinite suspension and loop adjunction, see Remark 5.12.

Example 2.30 The most basic example of a $*$ -module is the singleton $* \in \mathcal{LU}$. We have identifications $* \boxtimes_{\mathcal{L}} * = \mathcal{L}(2) \times_{\mathcal{L}(1)^2} (* \times *) \cong \mathcal{L}(2)/\mathcal{L}(1)^2$. The latter space consists only of a single point, as is proven in [5, Lemma I.8.1]. The above composition $* \boxtimes_{\mathcal{L}} * \rightarrow \mathcal{L}(2)/\mathcal{L}(1)^2 \cong *$ agrees with λ_* , hence $* \in \mathcal{M}_*$. More general, let $A \in \mathcal{U}$ be equipped with the trivial $\mathcal{L}(1)$ -action. Then

$$A \boxtimes_{\mathcal{L}} * = \mathcal{L}(2) \times_{\mathcal{L}(1)^2} (A \times *) \cong (\mathcal{L}(2) \times_{\mathcal{L}(1)^2} (* \times *)) \times A = (* \boxtimes_{\mathcal{L}} *) \times A \cong A$$

and λ is compatible with these isomorphisms. Thus, A is a $*$ -module.

Example 2.31 Let V be a G -representation of finite dimension, then $\mathbf{L}(V, \mathbb{R}^\infty)$ is an \mathcal{L} -space via postcomposition on \mathbb{R}^∞ and has a right G -action induced by the action on the representation V . The quotient space $\mathbf{L}(V, \mathbb{R}^\infty)/G$ is a $*$ -module, as will follow from Corollary 3.17.

Example 2.32 The spaces $\mathcal{L}(n)$ are never $*$ -modules if $n \geq 1$. As they are all isomorphic, it suffices to show this for $n = 1$, where λ is the map

$$\mathcal{L}(2) \times_{\mathcal{L}(1)^2} (\mathcal{L}(1) \times *) \rightarrow \mathcal{L}(1), \quad [\psi, \phi, *] \mapsto \psi_1 \circ \phi.$$

For each $\psi \in \mathcal{L}(2)$, the orthogonal complement $\text{im}(\psi_1)^\perp$ must be infinite-dimensional (such that the linear map $\psi = \psi_1 \oplus \psi_2: (\mathbb{R}^\infty)^2 \rightarrow \mathbb{R}^\infty$ can still be isometric), and precomposition by some element $\phi \in \mathcal{L}(1)$ does not affect this property. Consequently, maps $\theta \in \mathcal{L}(1)$ such that $\text{im}(\theta)^\perp$ is finite-dimensional cannot be in the image of $\lambda_{\mathcal{L}(1)}$.

Example 2.33 Let V_G be a G -representation of countably infinite dimension, then the same argument as in the previous example shows that $\mathbf{L}(V_G, \mathbb{R}^\infty)/G \notin \mathcal{M}_*$.

Analogous to $*$ -modules, which are unital \mathcal{L} -spaces, we can define the category of *counital* \mathcal{L} -spaces in a dual way in terms of the adjoint $\bar{\lambda}_X$.

Definition 2.34 An \mathcal{L} -space Y is a *counital \mathcal{L} -space* or *co- $*$ -module* if the natural map $\bar{\lambda}_Y: Y \rightarrow F_{\boxtimes_{\mathcal{L}}}(*, Y)$ is an isomorphism. Write $\mathcal{M}^* \subseteq \mathcal{LU}$ for the full subcategory of co- $*$ -modules.

Lemma 2.35 The functor $- \boxtimes_{\mathcal{L}} *: \mathcal{LU} \rightarrow \mathcal{LU}$ takes values in \mathcal{M}_* . The functor $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{LU} \rightarrow \mathcal{LU}$ takes values in \mathcal{M}^* .

Proof. Under the associativity isomorphism $(X \boxtimes_{\mathcal{L}} *) \boxtimes_{\mathcal{L}} * \cong X \boxtimes_{\mathcal{L}} (* \boxtimes_{\mathcal{L}} *)$, the map $\lambda_{X \boxtimes_{\mathcal{L}} *} \boxtimes_{\mathcal{L}} *$ agrees with $X \boxtimes_{\mathcal{L}} \lambda_*$, hence $X \boxtimes_{\mathcal{L}} *$ is a $*$ -module by Example 2.30. We have a chain of natural isomorphisms

$$\begin{aligned} \mathcal{LU}(X, F_{\boxtimes_{\mathcal{L}}}(*, F_{\boxtimes_{\mathcal{L}}}(*, Y))) &\cong \mathcal{LU}(X \boxtimes_{\mathcal{L}} *, F_{\boxtimes_{\mathcal{L}}}(*, Y)) \\ &\cong \mathcal{LU}(X \boxtimes_{\mathcal{L}} * \boxtimes_{\mathcal{L}} *, Y) \cong \mathcal{LU}(X \boxtimes_{\mathcal{L}} *, Y) \end{aligned}$$

where the map in the middle is induced by $\lambda_{X \boxtimes_{\mathcal{L}} *}$. By uniqueness of adjoints, $F_{\boxtimes_{\mathcal{L}}}(*, Y)$ and $F_{\boxtimes_{\mathcal{L}}}(*, F_{\boxtimes_{\mathcal{L}}}(*, Y))$, considered as functors in Y , are naturally equivalent via $\bar{\lambda}_Y$. Thus, $F_{\boxtimes_{\mathcal{L}}}(*, Y)$ is a co- $*$ -module for all $Y \in \mathcal{LU}$. \square

Lemma 2.36 The box product of an \mathcal{L} -space X with a $*$ -module Y is a $*$ -module.

Proof. Under the associativity isomorphism $(X \boxtimes_{\mathcal{L}} Y) \boxtimes_{\mathcal{L}} * \cong X \boxtimes_{\mathcal{L}} (Y \boxtimes_{\mathcal{L}} *)$, the map $\lambda_{X \boxtimes_{\mathcal{L}} Y} \boxtimes_{\mathcal{L}} *$ agrees with the isomorphism $X \boxtimes_{\mathcal{L}} \lambda_Y$. \square

The category \mathcal{M}^* will play a significant role in our attempt to establish model structures for \mathcal{M}_* . This is due to the fact that model structures can be transported along adjunctions associated to well-behaved monads. A reinterpretation of \mathcal{M}^* in terms of monads will be given in the next section. The following proposition then links unital and counital \mathcal{L} -spaces and identifies them as “mirror image subcategories”. This name is due to Elmendorf, Kriz, Mandell and May, who use it to describe a similar situation for S -modules, see [5, Sect. II.2].

Proposition 2.37 (cf. [1], Sect. 4.3) The categories \mathcal{M}_* and \mathcal{M}^* of unital and counital \mathcal{L} -spaces, respectively, are “mirror image subcategories” in the following sense:

- a) All pairs of functors in the diagram below form adjunctions (where upper arrows and arrows on the left hand side are left adjoints).

$$\begin{array}{ccc}
 \mathcal{LU} & \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}}^*} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(*, -)} \end{array} & \mathcal{M}_* \\
 \begin{array}{c} \uparrow \\ F_{\boxtimes_{\mathcal{L}}}(*, -) \\ \downarrow \end{array} & \begin{array}{c} \nearrow \\ F_{\boxtimes_{\mathcal{L}}}(*, -) \\ \searrow \end{array} & \begin{array}{c} \uparrow \\ -\boxtimes_{\mathcal{L}}^* \\ \downarrow \end{array} \\
 \mathcal{M}^* & \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}}^*} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(*, -)} \end{array} & \mathcal{LU}
 \end{array}$$

- b) The subdiagrams of left-adjoint (respectively right-adjoint) functors commute up to natural equivalence.
- c) The categories \mathcal{M}_* and \mathcal{M}^* are bicomplete. Colimits in \mathcal{M}_* are created in \mathcal{LU} , limits are obtained by applying $-\boxtimes_{\mathcal{L}}^*$ to a limit in \mathcal{LU} ; dually for \mathcal{M}^* .
- d) The diagonal adjunction (co-)restricts to an equivalence of categories

$$\mathcal{M}^* \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}}^*} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(*, -)} \end{array} \mathcal{M}_*.$$

Proof. Ad a): The diagonal adjunction was established in Lemma 2.21. (Co-)restriction yields the two horizontal adjunctions. It is easy to see that the vertical pair on the left is an adjunction with unit $\bar{\lambda}$ and counit $\bar{\lambda}^{-1}$, and that the vertical pair on the right is an adjunction with counit λ and unit λ^{-1} .

Ad b): Some parts of the commutativity statement are obvious; the others follow from abstract nonsense: Composition of the upper and right (respectively lower and left) adjunctions provides two new adjunctions,

$$\mathcal{LU} \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}}^*} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(*, * \boxtimes_{\mathcal{L}} -)} \end{array} \mathcal{LU} \quad \text{and} \quad \mathcal{LU} \begin{array}{c} \xrightarrow{F_{\boxtimes_{\mathcal{L}}}(*, -) \boxtimes_{\mathcal{L}}^*} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(*, -)} \end{array} \mathcal{LU}.$$

By uniqueness of adjoints, we see that

$$F_{\boxtimes_{\mathcal{L}}}(*, \lambda_Y) : F_{\boxtimes_{\mathcal{L}}}(*, * \boxtimes_{\mathcal{L}} Y) \rightarrow F_{\boxtimes_{\mathcal{L}}}(*, Y)$$

and

$$\bar{\lambda}_X \boxtimes_{\mathcal{L}}^* : X \boxtimes_{\mathcal{L}}^* \rightarrow F_{\boxtimes_{\mathcal{L}}}(*, X) \boxtimes_{\mathcal{L}}^*$$

are natural isomorphisms for all $X, Y \in \mathcal{LU}$.

Ad c): The category \mathcal{M}^* is a reflective subcategory and \mathcal{M}_* is a coreflective subcategory in the sense of [9, Sect. IV.3]. The statement about (co-)limits is a direct consequence.

Ad d): For $X \in \mathcal{M}_*$, $Y \in \mathcal{M}^*$, the natural isomorphisms from the proof of b) become natural isomorphisms $X \rightarrow * \boxtimes_{\mathcal{L}} F_{\boxtimes_{\mathcal{L}}}(*, X)$ and $F_{\boxtimes_{\mathcal{L}}}(*, * \boxtimes_{\mathcal{L}} Y) \rightarrow Y$. These are unit and counit of the desired equivalence of categories. \square

Corollary 2.38 Under the box product, \mathcal{M}_* is a closed symmetric monoidal category in the usual sense.

Proof. The box product of two $*$ -modules is a $*$ -module by Corollary 2.36, hence the box product functor is well-defined. It follows that \mathcal{M}_* is symmetric monoidal. For a $*$ -module Y , the functor $- \boxtimes_{\mathcal{L}} Y: \mathcal{M}_* \rightarrow \mathcal{M}_*$ factors as the composition of the inclusion $\mathcal{M}_* \rightarrow \mathcal{LU}$ and the box product functor $- \boxtimes_{\mathcal{L}} Y: \mathcal{LU} \rightarrow \mathcal{M}_*$. By Lemma 2.21 and Proposition 2.37, both have right adjoints, whose composition $F_{\boxtimes_{\mathcal{L}}}(Y, -) \boxtimes_{\mathcal{L}} *: \mathcal{M}_* \rightarrow \mathcal{M}_*$ is the desired right adjoint. \square

2.5. Reinterpretation in terms of monads

We show that the categories of \mathcal{L} -spaces and co- $*$ -modules are categories of algebras over certain monads \mathbb{L} and \mathbb{F} on the category of spaces. Recall that a monad is a monoid-like endofunctor.

Definition 2.39 (cf. [9], Sect. VI.1) A *monad* T on a category \mathcal{C} is an endofunctor $T: \mathcal{C} \rightarrow \mathcal{C}$ together with two natural transformations, the *unit* $\eta: \text{id}_{\mathcal{C}} \rightarrow T$, and the *multiplication* $\mu: T^2 \rightarrow T$, such that the following associativity and unitality diagrams commute:

$$\begin{array}{ccc}
 T^3 & \xrightarrow{T\mu} & T^2 \\
 \mu T \downarrow & & \downarrow \mu \\
 T^2 & \xrightarrow{\mu} & T
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \text{id}_{\mathcal{C}} \circ T & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & T \circ \text{id}_{\mathcal{C}} \\
 \text{id} \downarrow & & \downarrow \mu & & \downarrow \text{id} \\
 T & \xrightarrow{\text{id}} & T & \xleftarrow{\text{id}} & T
 \end{array}$$

Every adjoint pair of functors gives rise to a monad as follows. Let

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$$

be an adjunction. Define T to be the composite GF , let the monad unit η be the unit $\eta: \text{id}_{\mathcal{C}} \rightarrow GF = T$ of the adjunction, and define the multiplication in terms of the adjunction counit $\varepsilon: FG \rightarrow \text{id}_{\mathcal{D}}$ as $\mu := G\varepsilon F: T^2 = GF GF \rightarrow GF = T$. This data defines a monad.

Definition 2.40 (cf. [9], Sect. VI.2) An *algebra* over a monad $T: \mathcal{C} \rightarrow \mathcal{C}$ is an object $A \in \mathcal{C}$ together with a *structure map* $\alpha: TA \rightarrow A$ such that the following associativity and unitality diagrams commute:

$$\begin{array}{ccc}
 T^2A & \xrightarrow{T\alpha} & TA \\
 \mu_A \downarrow & & \downarrow \alpha \\
 TA & \xrightarrow{\alpha} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{\eta_A} & TA \\
 \searrow \text{id} & & \downarrow \alpha \\
 & & A
 \end{array}$$

Given a monad $T: \mathcal{C} \rightarrow \mathcal{C}$, write $\mathcal{C}[T]$ for its associated category of algebras. There is always a *free T -algebra functor* sending $X \in \mathcal{C}$ to the T -algebra TX with structure map μ_X given by the monad multiplication. It is left adjoint to the forgetful functor $\mathcal{C}[T] \rightarrow \mathcal{C}$. The monad T is determined by this adjunction. For details, see [9, Sect. VI.2].

Monads generalize group actions: For $\mathcal{C} = \mathbf{Set}$ and G a group, there is a monad with underlying functor $M \mapsto G \times M$ such that the associated algebras are nothing else but sets with a G -action in the usual sense. We shall spell out the details in the setting of spaces with monoid actions.

Example 2.41 Consider the endofunctor $\mathbb{L}: \mathcal{U} \rightarrow \mathcal{U}$ that sends $X \in \mathcal{U}$ to $\mathcal{L}(1) \times X$. Define natural transformations

$$\eta_X: X \rightarrow \mathcal{L}(1) \times X, \quad x \mapsto (\text{id}, x)$$

and

$$\mu_X: \mathcal{L}(1) \times \mathcal{L}(1) \times X \rightarrow \mathcal{L}(1) \times X, \quad (\phi_1, \phi_2, x) \mapsto (\phi_1\phi_2, x).$$

One readily verifies that \mathbb{L} is a monad on \mathcal{U} and that the notion of an \mathbb{L} -algebra agrees with the notion of an \mathcal{L} -space. Thus, we obtain an isomorphism of categories $\mathcal{LU} \cong \mathcal{U}[\mathbb{L}]$.

Not only \mathcal{LU} , but also \mathcal{M}^* can be described as a category of algebras over a monad.

Example 2.42 Recall that $F = F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{LU} \rightarrow \mathcal{M}^*$ is left adjoint to the inclusion functor $R: \mathcal{M}^* \rightarrow \mathcal{LU}$. The associated monad $\mathbb{F} = RF$ is just the functor $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{LU} \rightarrow \mathcal{LU}$.

The names F and R follow the notation of [1, Lemma 4.9] except that we decided to use upper case letters for the functors to avoid confusion with morphisms.

Proposition 2.43 (cf. [1], Sect. 4.3) The category of algebras over \mathbb{F} is isomorphic to the category \mathcal{M}^* of counital \mathcal{L} -spaces.

Proof. This is completely analogous to the proof of [5, Prop. II.2.7]. \square

2.6. Equivalences of \mathcal{L} -spaces

We introduce various notions of an equivalence of \mathcal{L} -spaces, and give some basic examples of model structures for \mathcal{LU} . We begin with the naïve approach of non-equivariant equivalences and forget about $\mathcal{L}(1)$ -actions for a moment. However, as we have taken the perspective of global homotopy theory, the main part of this section is devoted to the discussion of equivariant equivalences.

Call a morphism of \mathcal{L} -spaces $f: X \rightarrow Y$ a weak homotopy equivalence (respectively Serre fibration) if the underlying map of spaces has this property. A first model structure for \mathcal{LU} can now be derived as a special case of Theorem A.30 which we discuss more generally in the appendix. It guarantees the existence of a model structure in which the weak equivalences and fibrations can be detected on N -fixed points for all N in a suitable collection of submonoids of $\mathcal{L}(1)$. Choose this family to be the trivial family consisting only of the trivial subgroup $\{\text{id}\} \leq \mathcal{L}(1)$ to obtain the next theorem.

Theorem 2.44 (Non-equivariant model structure for \mathcal{LU}) The weak equivalences and Serre fibrations form a proper topological model structure on \mathcal{LU} . It is cofibrantly generated with sets of generating (acyclic) cofibrations I_{non} (respectively J_{non}) as follows:

$$I_{\text{non}} = \{\mathcal{L}(1) \times i_k^{\mathbb{Q}}\}, \quad J_{\text{non}} = \{\mathcal{L}(1) \times j_k^{\mathbb{Q}}\}$$

We write $(\mathcal{LU})_{\text{non}}$ for the category of \mathcal{L} -spaces equipped with the non-equivariant model structure.

Remark 2.45 The existence of the non-equivariant model structure can also be deduced from a “monadic” point of view by lifting the Quillen model structure for spaces to \mathcal{LU} , see Section 5.2. It first appeared in [1, Sect. 4.6].

The non-equivariant model structure is compatible with the box product, as captured by the following definition. Recall that a weak symmetric monoidal category was introduced in Definition 2.26. We remind the reader of the pushout product axiom as stated in [14, Def. 3.1].

Pushout product axiom. Let $f: X \rightarrow Y$ and $f': X' \rightarrow Y'$ be cofibrations in a model category \mathcal{C} which admits a biproduct \otimes . Then the *pushout product*

$$f \square f': X \otimes Y' \cup_{X \otimes X'} Y \otimes X' \rightarrow Y \otimes Y'$$

of f and f' is a cofibration, too. If in addition f or f' is a weak equivalence, then so is the pushout product. \square

Definition 2.46 Let \mathcal{C} be a model category with weak equivalences the class of morphisms \mathcal{W} . Assume in addition that $(\mathcal{C}, \mathcal{W})$ is a weak symmetric monoidal category under the biproduct \otimes . Then \mathcal{C} is called a *weak symmetric monoidal model category* if it satisfies the pushout product axiom.

Our definition is simply the “weak” analogue of the usual definition.

Remark 2.47 Some authors require an extra condition in their definition of a symmetric monoidal model category to ensure that the unit object, which is not cofibrant in general, is a unit object with respect to the derived biproduct on the homotopy category level, cf. [14, Rem. 3.2]. A sufficient condition is that for any cofibrant replacement $\text{cof}(\mathbb{1}) \rightarrow \mathbb{1}$ of the unit object, the natural map $X \otimes \text{cof}(\mathbb{1}) \rightarrow X \otimes \mathbb{1}$ is a weak equivalence for all cofibrant objects $X \in \mathcal{C}$. We do not require that such a condition holds because we will never work in the homotopy category.

The reader interested in the derived products should note that both functors $X \times -$ and $X \boxtimes_{\mathcal{L}} -$ preserve all kinds of equivalences defined in terms of fixed points, so the extra condition will usually be satisfied.

The following computation of the pushout product will be useful.

Lemma 2.48 The pushout product of two \mathcal{L} -maps $X \times i_k^{\mathcal{Q}}: X \times S^{k-1} \rightarrow X \times D^k$ and $Y \times i_m^{\mathcal{Q}}: Y \times S^{m-1} \rightarrow Y \times D^m$ with respect to the box product is isomorphic to the \mathcal{L} -map $(X \boxtimes_{\mathcal{L}} Y) \times i_{k+m}^{\mathcal{Q}}: (X \boxtimes_{\mathcal{L}} Y) \times S^{k+m-1} \rightarrow (X \boxtimes_{\mathcal{L}} Y) \times D^{k+m}$.

Proof. For all trivial \mathcal{L} -spaces A and B , we have

$$(X \times A) \boxtimes_{\mathcal{L}} (Y \times B) \cong (X \boxtimes_{\mathcal{L}} Y) \times (A \times B).$$

As the left adjoint $(X \boxtimes_{\mathcal{L}} Y) \times (-): \mathcal{L}\mathcal{U} \rightarrow \mathcal{L}\mathcal{U}$ preserves colimits, it suffices to compute the pushout product of $i_k^{\mathcal{Q}}$ and $i_m^{\mathcal{Q}}$ with respect to the cartesian product, but up to homeomorphism, this is just $i_{k+m}^{\mathcal{Q}}$. \square

Proposition 2.49 The category of \mathcal{L} -spaces is a weak symmetric monoidal model category with respect to the box product and the non-equivariant model structure.

Proof. It suffices to check the pushout product axiom for generating cofibrations and acyclic cofibrations, cf. [14, Lemma 3.5]. By the lemma above, the pushout product of generating cofibrations $\mathcal{L}(1) \times i_k^{\mathcal{Q}}$ and $\mathcal{L}(1) \times i_m^{\mathcal{Q}}$ is isomorphic to the \mathcal{L} -map

$$(\mathcal{L}(1) \boxtimes_{\mathcal{L}} \mathcal{L}(1)) \times i_{k+m}^{\mathcal{Q}}.$$

Since we know that $\mathcal{L}(1) \boxtimes_{\mathcal{L}} \mathcal{L}(1) \cong \mathcal{L}(2) \cong \mathcal{L}(1)$, see Examples 2.6 and 2.16, the pushout product of two generating cofibrations is a cofibration.

Similarly, the pushout product of a generating cofibration $i_k^{\mathcal{Q}}$ and a generating acyclic cofibration $j_m^{\mathcal{Q}}$ is isomorphic to $\mathcal{L}(1) \times (i_k^{\mathcal{Q}} \square j_m^{\mathcal{Q}})$ where $(i_k^{\mathcal{Q}} \square j_m^{\mathcal{Q}})$ is the pushout product with respect to the cartesian product, which is a weak homotopy equivalence. Of course, $\mathcal{L}(1) \times -: \mathcal{L}\mathcal{U} \rightarrow \mathcal{L}\mathcal{U}$ preserves weak equivalences. \square

There are several ways of defining an equivalence of \mathcal{L} -spaces in an equivariant fashion. First of all, $\mathcal{L}\mathcal{U}$ comes with a notion of \mathcal{L} -homotopy equivalence as it is tensored over spaces, see Section A.3 for details. We will also need the following notions of equivalences (and fibrations).

Definition 2.50 We call a morphism $f: X \rightarrow Y$ of \mathcal{L} -spaces a

- 1) \mathfrak{C} -equivalence (respectively a \mathfrak{C} -fibration) if the induced map $f^G: X^G \rightarrow Y^G$ on G -fixed points is a weak homotopy equivalence (respectively a Serre fibration) for all \mathfrak{C} -subgroups $G \leq \mathcal{L}(1)$.

- 2) \mathfrak{F} -equivalence (respectively an \mathfrak{F} -fibration) if the induced map $f^G: X^G \rightarrow Y^G$ on G -fixed points is a weak homotopy equivalence (respectively a Serre fibration) for all \mathfrak{F} -subgroups $G \leq \mathcal{L}(1)$.
- 3) *strong \mathfrak{C} -equivalence* if the underlying map of G -spaces is a G -homotopy equivalence for all \mathfrak{C} -subgroups $G \leq \mathcal{L}(1)$.
- 4) *strong \mathfrak{F} -equivalence* if the underlying map of G -spaces is a G -homotopy equivalence for all \mathfrak{F} -subgroups $G \leq \mathcal{L}(1)$.

All kinds of equivalences of $*$ -modules will be defined on underlying \mathcal{L} -spaces, e.g. we will call a map in \mathcal{M}_* a \mathfrak{C} -equivalence if it is a \mathfrak{C} -equivalence under the forgetful functor $\mathcal{M}_* \rightarrow \mathcal{LU}$.

We will frequently refer to the (strong) \mathfrak{C} -equivalences as (*strong*) *global equivalences*. This is justified by Lemma 3.27.

A strong \mathfrak{F} -equivalence may not be an \mathcal{L} -homotopy equivalence in general because the data of the different homotopy inverses and homotopies depending on the groups G might not be enough to define a homotopy inverse which is $\mathcal{L}(1)$ -equivariant.

Lemma 2.51 We have the following obvious implications:

$$\begin{array}{ccccc}
\mathcal{L}\text{-homotopy equiv.} & \Rightarrow & \text{strong } \mathfrak{F}\text{-equiv.} & \Rightarrow & \mathfrak{F}\text{-equiv.} \\
& & \Downarrow & & \Downarrow \\
& & \text{strong global equiv.} & \Rightarrow & \text{global equiv.} \\
& & \Downarrow & & \Downarrow \\
& & G\text{-homotopy equiv.} & \Rightarrow & G\text{-weak equiv.} \\
& & \Downarrow & & \Downarrow \\
& & \text{homotopy equiv.} & \Rightarrow & \text{weak equiv.}
\end{array}$$

Example 2.52 (cf. [13], I.7.14) Let $G \leq \mathcal{L}(1)$ be an \mathfrak{F} -subgroup and $\iota: V \rightarrow \mathbb{R}_G^\infty$ the inclusion of a faithful finite-dimensional G -subrepresentation. Then the induced map

$$\iota^*/G: \mathcal{L}/G = \mathbf{L}(\mathbb{R}_G^\infty, \mathbb{R}^\infty)/G \rightarrow \mathbf{L}(V, \mathbb{R}^\infty)/G$$

is a strong \mathfrak{F} -equivalence of \mathcal{L} -spaces.

Remark 2.53 The statements of [13, I.2.10, I.7.12, I.7.14, I.7.22], including the preceding example, are only proven for \mathfrak{C} -subgroups. Nevertheless, the same proofs still

work for faithful universes (respectively \mathfrak{F} -subgroups), except for the very last statement of I.2.10 ii), whose proof fails in the faithful case because global equivalences of orthogonal spaces (see Section 3.3) cannot be detected on faithful universes.

Corollary 2.54 Let G be a compact Lie group and $\phi: V_G \rightarrow U_G$ a G -equivariant inclusion of faithful G -universes. Then the induced map

$$\phi^*/G: \mathbf{L}(U_G, \mathbb{R}^\infty)/G \rightarrow \mathbf{L}(V_G, \mathbb{R}^\infty)/G$$

is a strong \mathfrak{F} -equivalence of \mathcal{L} -spaces.

Proof. Choose a finite-dimensional faithful subrepresentation $V \subset V_G$, then the inclusions $V \rightarrow V_G$ and $V \rightarrow U_G$ induce strong \mathfrak{F} -equivalences by Example 2.52. Thus, the inclusion ϕ must induce a strong \mathfrak{F} -equivalence by the 2-out-of-3 property. \square

When studying the point-set properties of the operad \mathcal{L} , Elmendorf, Kriz, Mandell and May proved that the unital transformation $\lambda_X: X \boxtimes_{\mathcal{L}} * \rightarrow X$ of the box product is a homotopy equivalence for $X = \mathcal{L}(1)$, see [5, Lemma I.8.4]. In fact, it is a weak homotopy equivalence for all $X \in \mathcal{LU}$, as can be proven similarly to [5, Thm. I.8.5], cf. [1, Sect. 4.1]. A much stronger statement was proven by Schwede.

Theorem 2.55 ([13], Thm. I.7.22) The natural map $\lambda_{X,Y}: X \boxtimes_{\mathcal{L}} Y \rightarrow X \times Y$ of \mathcal{L} -spaces from 2.22 is a strong \mathfrak{F} -equivalence for all $X, Y \in \mathcal{LU}$. As a direct consequence, the unital transformation $\lambda_X: X \boxtimes_{\mathcal{L}} * \rightarrow X$ is a strong \mathfrak{F} -equivalence for all $X \in \mathcal{LU}$.

We collect some formal consequences that will be important for the proof of our main theorem.

Corollary 2.56 The natural map $\bar{\lambda}_X: X \rightarrow \mathbf{F}_{\boxtimes_{\mathcal{L}}}(*, X)$ is a strong \mathfrak{F} -equivalence for all \mathcal{L} -spaces X .

Proof. Consider the following commutative diagram and use the fact that strong \mathfrak{F} -equivalences have the 2-out-of-3 property:

$$\begin{array}{ccc} X \boxtimes_{\mathcal{L}} * & \xrightarrow[\sim]{\lambda_X} & X \\ \bar{\lambda}_X \boxtimes_{\mathcal{L}} * \downarrow \cong & & \downarrow \bar{\lambda}_X \\ \mathbf{F}_{\boxtimes_{\mathcal{L}}}(*, X) \boxtimes_{\mathcal{L}} * & \xrightarrow[\sim]{\lambda_{\mathbf{F}_{\boxtimes_{\mathcal{L}}}(*, X)}} & \mathbf{F}_{\boxtimes_{\mathcal{L}}}(*, X) \end{array}$$

The left arrow is the natural isomorphism appearing in the proof of part b) of Proposition 2.37. The horizontal arrows are strong \mathfrak{F} -equivalences by the preceding theorem. \square

Corollary 2.57 The functors $-\boxtimes_{\mathcal{L}}*$, $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{LU} \rightarrow \mathcal{LU}$ preserve and reflect strong \mathfrak{F} -equivalences and all the weaker notions of equivalence from Lemma 2.51.

Proof. Use the 2-out-of-3 property again:

$$\begin{array}{ccc}
 X \boxtimes_{\mathcal{L}} * & \xrightarrow{f \boxtimes_{\mathcal{L}} *} & Y \boxtimes_{\mathcal{L}} * \\
 \lambda_X \downarrow & & \downarrow \lambda_Y \\
 X & \xrightarrow{f} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \bar{\lambda}_X \downarrow & & \downarrow \bar{\lambda}_Y \\
 F_{\boxtimes_{\mathcal{L}}}(*, X) & \xrightarrow{F_{\boxtimes_{\mathcal{L}}}(*, f)} & F_{\boxtimes_{\mathcal{L}}}(*, Y)
 \end{array}$$

\square

Corollary 2.58 Let $Z \in \mathcal{LU}$ be non-empty. The functor $-\boxtimes_{\mathcal{L}} Z: \mathcal{LU} \rightarrow \mathcal{LU}$ preserves strong \mathfrak{F} -equivalences and all the weaker notions of equivalence from Lemma 2.51. It reflects \mathfrak{F} -equivalences, \mathfrak{C} -equivalences, homotopy equivalences and weak homotopy equivalences.

Proof. Let $g: X \rightarrow Y$ be a map of \mathcal{L} -spaces. As $\lambda_{W,Z}: W \boxtimes_{\mathcal{L}} Z \rightarrow W \times Z$ is a natural strong \mathfrak{F} -equivalence, it suffices to prove the desired properties for $-\times Z: \mathcal{LU} \rightarrow \mathcal{LU}$. First assume that G is a compact Lie group and g has a G -homotopy inverse $h: Y \rightarrow X$, then $g \times \text{id}_Z$ and $h \times \text{id}_Z$ are mutual G -homotopy inverses, too. This proves that $-\times Z$ preserves strong \mathfrak{C} - and \mathfrak{C} -equivalences as well as ordinary G -homotopy equivalences. Now we show that g is a G -weak equivalence if and only if $g \times Z$ is. Taking G -fixed points commutes with products, hence it is enough to show that g^G is a weak homotopy equivalence if and only if $g^G \times Z^G$ is. This is true for all spaces because homotopy groups also commute with products. It follows that $-\times Z$ preserves and reflects \mathfrak{F} -equivalences and \mathfrak{C} -equivalences, and also weak homotopy equivalences if we choose G to be the trivial group.

We still have to show that the functor reflects homotopy equivalences of underlying spaces. If $g \times Z$ is a homotopy equivalence, then any choice of a basepoint $z \in Z$ yields a section for the constant map $Z \rightarrow *$ and exhibits g as a retract of $g \times Z$, which must be a homotopy equivalence. \square

Note that G -homotopy equivalences, and consequently strong \mathfrak{F} - and \mathfrak{C} -equivalences, are not reflected in general since there may be maps $f: X \rightarrow Y$ of G -spaces such that $f \times Z: X \times Z \rightarrow Y \times Z$ is a G -homotopy equivalence but f is not.

As another special case of Theorem A.30, we obtain our first two examples of equivariant model structures where the weak equivalences are the \mathfrak{C} - or \mathfrak{F} -equivalences. This time, we choose the collection of biclosed submonoids to be the family $\mathcal{C}_{\mathfrak{C}}$ of all completely universal subgroups, or the family $\mathcal{C}_{\mathfrak{F}}$ of all faithfully universal subgroups, respectively.

Proposition 2.59 (\mathfrak{C} -projective model structure for \mathcal{LU} , [13], Prop. I.7.15) The global equivalences and global fibrations define a cofibrantly generated proper topological model structure on \mathcal{LU} . Sets of generating (acyclic) cofibrations are given by $I_{\mathfrak{C}}$ (respectively $J_{\mathfrak{C}}$), where

$$I_{\mathfrak{C}} = \{\mathcal{L}/G \times i_k^{\mathcal{Q}}\}, \quad J_{\mathfrak{C}} = \{\mathcal{L}/G \times j_k^{\mathcal{Q}}\}$$

for all \mathfrak{C} -subgroups $G \leq \mathcal{L}(1)$. Write $(\mathcal{LU})_{\mathfrak{C}}$ for the category of \mathcal{L} -spaces equipped with the \mathfrak{C} -projective global model structure.

Proposition 2.60 (\mathfrak{F} -projective model structure for \mathcal{LU}) The analogous statement with respect to the family $\mathcal{C}_{\mathfrak{F}}$ of faithfully universal subgroups is true. Write $(\mathcal{LU})_{\mathfrak{F}}$ for the category of \mathcal{L} -spaces equipped with the \mathfrak{F} -projective global model structure.

In accordance with [13], we will sometimes refer to $(\mathcal{LU})_{\mathfrak{C}}$ as the *projective global model structure* if it is clear from the context that we restrict ourselves to completely universal subgroups. As we agreed that the global equivalences shall be the \mathfrak{C} -equivalences, the projective model structure is a *global* model structure in the sense that the global equivalences form the class of weak equivalences. We will see more examples of global model categories of \mathcal{L} -spaces in Chapter 4.

Example 2.61 Observe that the singleton $*$ $\in \mathcal{LU}$ is not cofibrant in the \mathfrak{F} -projective nor in the \mathfrak{C} -projective model structure: Every object is fibrant, and the restriction map $\mathcal{L}(1) \rightarrow \mathbf{L}(V, \mathbb{R}^{\infty})$ for a faithful finite-dimensional subrepresentation $V \subseteq \mathbb{R}^{\infty}$ is an \mathfrak{F} -equivalence by Example 2.52. In particular (for $V = \{0\}$), the constant map $\mathcal{L}(1) \rightarrow *$ is an acyclic \mathfrak{F} -fibration (and hence an acyclic \mathfrak{C} -fibration). But the lifting problem

$$\begin{array}{ccc}
\emptyset & \longrightarrow & \mathcal{L}(1) \\
\downarrow & \nearrow & \downarrow \\
* & \longrightarrow & *
\end{array}$$

cannot admit a solution because there are no maps from a trivial \mathcal{L} -space to one without $\mathcal{L}(1)$ -fixed points.

The \mathfrak{F} -projective model structure is compatible with the box product, while the \mathfrak{C} -projective model structure does not have this property due to Example 2.17. We will show in Section 4.3 that there is a global model structure for $\mathcal{L}\mathcal{U}$ which is indeed weakly symmetric monoidal.

Proposition 2.62 The category of \mathcal{L} -spaces is a weak symmetric monoidal model category with respect to the box product and the \mathfrak{F} -projective model structure.

Proof. Consider two generating cofibrations $\mathcal{L}/G \times i_k^{\mathcal{Q}}$ and $\mathcal{L}/K \times i_m^{\mathcal{Q}}$ for the \mathfrak{F} -projective model structure, where $G, K \leq \mathcal{L}(1)$ are \mathfrak{F} -subgroups. Our computation in Lemma 2.48 shows that their pushout product is isomorphic to $(\mathcal{L}/G \boxtimes_{\mathcal{L}} \mathcal{L}/K) \times i_{k+m}^{\mathcal{Q}}$, and by Example 2.17, the box product of the two quotient spaces can be identified with a space \mathcal{L}/H such that H is an \mathfrak{F} -subgroup isomorphic to $G \times K$. Thus, the pushout product of two generating cofibrations is isomorphic to a generating cofibration, and a similar reasoning applies if one of the maps is replaced by a generating acyclic cofibration. \square

3. Other models for global homotopy theory

This chapter introduces two categories of diagram spaces that serve as alternative models to capture the idea of global homotopy theory: the category \mathcal{IU} of *orthogonal spaces* and the category $O_{\text{gl}}\mathcal{U}$ of *orbispaces*. They complement our exposition of \mathcal{LU} . These models are related by adjoint pairs of functors as follows:

$$\mathcal{IU} \begin{array}{c} \xrightarrow{(-)(\mathbb{R}^\infty)} \\ \xleftarrow{u} \end{array} \mathcal{LU} \begin{array}{c} \xleftarrow{\Lambda} \\ \xrightarrow{\Phi} \end{array} O_{\text{gl}}\mathcal{U}$$

We first give a summary of the well-developed theory of orthogonal spaces which are functors from inner product spaces to \mathcal{U} . The category \mathcal{IU} comes with a box product that is symmetric monoidal in the usual sense. There is a notion of a global equivalence of orthogonal spaces which corresponds to the \mathcal{C} -equivalences of \mathcal{L} -spaces, and \mathcal{IU} admits a symmetric monoidal global model structure. We discuss how this global model structure relates to the two projective model structures for \mathcal{LU} and show that orthogonal spaces can be used to construct many examples of $*$ -modules.

In the last section, we introduce orbispaces, which come in two flavours with respect to the \mathcal{C} - or \mathfrak{F} -subgroups of $\mathcal{L}(1)$, respectively. In both cases, they are space-valued functors on a *global orbit category* such that a global analogue of Elmendorf's theorem holds.

3.1. Orthogonal spaces

We define orthogonal spaces and some of their properties, study some examples and establish the adjunction to \mathcal{L} -spaces.

Definition 3.1 Let \mathcal{I} be the category of finite-dimensional real inner product spaces with morphisms the linear isometric embeddings.

By Definition 2.2, \mathcal{I} is enriched over \mathcal{U} . We continue to write $\mathbf{L}(V, W)$ instead of $\mathcal{I}(V, W)$ for the space of linear isometric embeddings.

Definition 3.2 An *orthogonal space* is a continuous functor $Y: \mathcal{I} \rightarrow \mathcal{U}$. We write \mathcal{IU} for the category of orthogonal spaces and natural transformations.

Remark 3.3 Our notation concerning orthogonal spaces agrees with the one from Lind’s exposition of diagram spaces in [8]. The reader is warned that it differs from the notation in [13]: Schwede writes \mathbf{L} for the category \mathcal{I} and spc for the category \mathcal{IU} . In this thesis, we will always write \mathcal{JU} for the category of diagram spaces indexed on a topological category \mathcal{J} , and we refrain from writing \mathbf{LU} for orthogonal spaces to avoid confusion with \mathcal{LU} .

It is well-known that any functor category with values in a bicomplete category is itself bicomplete, where (co-)limits can be computed objectwise. In addition, we also have “weighted” colimits: The category \mathcal{IU} is tensored over \mathcal{U} where, for $Y \in \mathcal{IU}$, $A \in \mathcal{U}$, the tensor space $Y \times A$ sends $V \in \mathcal{I}$ to $(Y \times A)(V) := Y(V) \times A$. Equivalently, we can regard A as the constant orthogonal space with value A and form the product in \mathcal{IU} .

We explain why orthogonal spaces are another model for global homotopy theory: Let $Y \in \mathcal{IU}$ and assume that $V \in \mathcal{I}$ is a finite-dimensional G -representation (as always orthogonal). By functoriality of Y , the space $Y(V)$ is a G -space. Recall that we write $s(U_G)$ for the poset of finite-dimensional G -subrepresentations of a G -universe U_G , ordered by inclusions.

Definition 3.4 Let U_G be a complete G -universe (see Definition 2.7) and $Y \in \mathcal{IU}$. Define

$$Y(U_G) := \operatorname{colim}_{V \in s(U_G)} Y(V)$$

where the colimit is taken in the category \mathcal{GU} of G -spaces.

In [13, Sect. I.1], Schwede fixes a complete universe U_G for each compact Lie group and refers to $Y(U_G)$ as the *underlying G -space* of Y . Restricting attention to the collection $\{Y(U_G)\}$, we see that the data of an orthogonal space Y includes choices of G -spaces for all compact Lie groups G which are compatible in the sense that their underlying spaces arise from the same inner product spaces V and corresponding spaces $Y(V)$.

Remark 3.5 Orthogonal spaces are related to the theory of classifying spaces for families of groups: For a faithful G -representation V of finite dimension, the space $\mathbf{L}(V, U_K)$ is a universal space for the “family of graph subgroups” $\mathcal{F}(K; G^{op})$, and the “global classifying space” $\mathbf{L}(V, -)/G$ “universally represents G -bundles”, i.e., $\mathbf{L}(V, U_K)/G$ is a classifying space for principal G -bundles in the category of K -spaces. See [13, Sect. I.2] for details.

Example 3.6 We collect some examples of orthogonal spaces:

- 1) Let $A \in \mathcal{U}$ be a space, then there is the *constant orthogonal space* sending each $V \in \mathcal{I}$ to A and each linear isometric embedding to the identity.
- 2) Let G be a compact Lie group and V a G -representation of finite dimension. The *free orthogonal space* generated by $A \in G\mathcal{U}$ at (G, V) is the functor $\mathbf{L}_{G,V}A$ such that

$$(\mathbf{L}_{G,V}A)(W) = \mathbf{L}(V, W) \times_G A.$$

We briefly write $\mathbf{L}_{G,V}$ for the free orthogonal space $\mathbf{L}_{G,V}*$ that is generated by the singleton $* \in \mathcal{U}$. Note that the assignment $\mathbf{L}_{G,V}(-)$ is functorial itself.

Lemma 3.7 (cf. [13], I.2.5) The free orthogonal space functor $\mathbf{L}_{G,V}(-): G\mathcal{U} \rightarrow \mathcal{IU}$ is a left adjoint for the evaluation functor $\text{ev}_{G,V}: \mathcal{IU} \rightarrow G\mathcal{U}$ sending an orthogonal space Y to its value $Y(V)$ at the G -representation $V \in \mathcal{I}$.

It is not hard to see that free orthogonal spaces send all linear isometric embeddings to closed embeddings of spaces, see [13, Ex. I.2.7]. Thus, $Y(U_G)$ can be computed in the category **Top** of all topological spaces.

Each orthogonal space Y gives rise to an \mathcal{L} -space as follows: Evaluation of Y at \mathbb{R}^∞ , regarded as a (necessarily complete) universe for the trivial group, yields a space $Y(\mathbb{R}^\infty)$, and the following lemma guarantees that this assignment gives rise to a functor $(-)(\mathbb{R}^\infty): \mathcal{IU} \rightarrow \mathcal{LU}$.

Lemma 3.8 (cf. [13], Prop. I.7.2) Let Y be an orthogonal space. For $V \in \mathcal{I}$, there is a continuous “action” map $\mathbf{L}(V, \mathbb{R}^\infty) \times Y(V) \rightarrow Y(\mathbb{R}^\infty)$ sending (θ, y) to the value of y under the composition

$$Y(V) \xrightarrow{Y(\theta)} Y(\theta(V)) \xrightarrow{\text{can}} Y(\mathbb{R}^\infty)$$

where the first map is induced by a corestriction of θ to its image, and *can* is the canonical map to the colimit. These “action” maps assemble into a continuous $\mathcal{L}(1)$ -action on $Y(\mathbb{R}^\infty)$ in the colimit.

Proposition 3.9 The functor $(-)(\mathbb{R}^\infty): \mathcal{IU} \rightarrow \mathcal{LU}$ has a right adjoint $u: \mathcal{LU} \rightarrow \mathcal{IU}$. Its value for $X \in \mathcal{LU}$ on an inner product space $V \subseteq \mathbb{R}^\infty$ is the subspace

$$(uX)(V) = \{x \in X \mid \phi \cdot x = x \text{ for all } \phi \in \mathcal{L}(1) \text{ such that } \phi|_V = \text{id}\}.$$

In order to provide a convenient language for the proof, we define another family of submonoids of $\mathcal{L}(1)$.

Definition 3.10 Let $G \leq \mathbb{R}^\infty$ be an \mathfrak{F} -subgroup of $\mathcal{L}(1)$ and let $\iota: V \rightarrow \mathbb{R}^\infty$ be the inclusion of a faithful finite-dimensional subrepresentation of $\mathbb{R}_\mathbb{C}^\infty$. We define $\mathcal{L}[G, V]$ to be the stabilizer

$$\mathcal{L}[G, V] := \text{stab}_{\mathcal{L}(1)}(\iota G)$$

of the orbit of the inclusion $\iota G \in \mathbf{L}(V, \mathbb{R}^\infty)/G$.

The explicit description of the space $(uX)(V)$ shows that it is just the fixed point space $X^{\mathcal{L}[\{\text{id}\}, V]}$. As shown in [13, Construction I.7.16], the monoid $\mathcal{L}[G, V]$ contains the group G and there is an isomorphism of \mathcal{L} -spaces

$$\mathcal{L}/\mathcal{L}[G, V] \cong \mathbf{L}(V, \mathbb{R}^\infty)/G = \mathbf{L}_{G, V}(\mathbb{R}^\infty)$$

which is induced by the “action map”

$$\mathcal{L}(1) \rightarrow \mathbf{L}(V, \mathbb{R}^\infty)/G, \quad \phi \mapsto \phi \iota G.$$

Proof of Proposition 3.9. Assume for the moment that the right adjoint u exists. Then there are natural isomorphisms

$$(uX)(V) \cong \mathcal{U}(*, (uX)(V)) \cong \mathcal{I}\mathcal{U}(\mathbf{L}_{\{\text{id}\}, V}, uX) \cong \mathcal{L}\mathcal{U}(\mathbf{L}_{\{\text{id}\}, V}(\mathbb{R}^\infty), X) \cong X^{\mathcal{L}[\{\text{id}\}, V]}$$

which yield our description of $(uX)(V)$. For each $W \in \mathcal{I}$, choose a linear isometric embedding $\phi_W: W \rightarrow \mathbb{R}^\infty$ which we require to be the inclusion if $W \subseteq \mathbb{R}^\infty$. The assignment

$$(uX)(W) = (uX)(\phi_W(W)) = X^{\mathcal{L}[\{\text{id}\}, \phi_W(W)]}$$

defines a functor $u: \mathcal{L}\mathcal{U} \rightarrow \mathcal{I}\mathcal{U}$. Now let $Y \in \mathcal{I}\mathcal{U}$, $X \in \mathcal{L}\mathcal{U}$. The adjunction counit $\varepsilon: Y \rightarrow u(Y(\mathbb{R}^\infty))$ is induced on $V \in \mathcal{I}$ by the canonical map $Y(V) \rightarrow Y(\mathbb{R}^\infty)$ that factors over the subspace $(u(Y(\mathbb{R}^\infty)))(V)$. The inclusions $(uX)(V) \rightarrow X$ give rise to the adjunction unit $(uX)(\mathbb{R}^\infty) \rightarrow X$, and it is routine to verify the triangular identities. \square

3.2. The box product of orthogonal spaces

The category $\mathcal{I}\mathcal{U}$ of orthogonal spaces admits the structure of a symmetric monoidal category with respect to the so-called *box product*. It is a special case of the *Day convolution product* for functor categories. We do not attempt to cover the general theory

which can be found in [13, Sect. A.2]. Our treatment of the box product follows [13, Sect. I.3] closely.

Definition 3.11 A *bimorphism* $b: (X, Y) \rightarrow Z$ of orthogonal spaces $X, Y, Z \in \mathcal{IU}$ is a collection of $(\mathrm{O}(V) \times \mathrm{O}(W))$ -equivariant maps $b_{V,W}: X(V) \times Y(W) \rightarrow Z(V \oplus W)$ satisfying the following bilinearity condition: For all linear isometric embeddings $\varphi: V \rightarrow V'$, $\psi: W \rightarrow W'$ in \mathcal{I} , the following square commutes:

$$\begin{array}{ccc} X(V) \times Y(W) & \xrightarrow{b_{V,W}} & Z(V \oplus W) \\ X(\varphi) \times Y(\psi) \downarrow & & \downarrow Z(\varphi \oplus \psi) \\ X(V') \times Y(W') & \xrightarrow{b_{V',W'}} & Z(V' \oplus W') \end{array}$$

Definition 3.12 A *box product* of X and Y is an orthogonal space $X \boxtimes Y$ together with a bimorphism $i: (X, Y) \rightarrow X \boxtimes Y$ which is universal in the sense that for all $Z \in \mathcal{IU}$, the map $\mathcal{IU}(X \boxtimes Y, Z) \rightarrow \mathrm{Bimor}((X, Y), Z)$, $f \mapsto \{f(V \oplus W) \circ i_{V,W}\}_{V,W}$ is a bijection.

Example 3.13 The box product can be compared to the categorical product as follows: For orthogonal spaces X and Y in \mathcal{IU} , there is a bimorphism $(X, Y) \rightarrow X \times Y$ such that the maps $X(V) \times Y(W) \rightarrow X(V \oplus W) \times Y(V \oplus W) = (X \times Y)(V \oplus W)$ are induced by the two inclusions to the direct sum. This bimorphism corresponds to a map $\Lambda_{X,Y}: X \boxtimes Y \rightarrow X \times Y$ of orthogonal spaces. We will see later that it is always a weak equivalence in the global model structure of orthogonal spaces.

Example 3.14 (cf. [13], I.3.2) The box product of free orthogonal spaces is again free. More precisely, there is a natural isomorphism

$$(\mathbf{L}_{G,V}A) \boxtimes (\mathbf{L}_{K,W}B) \cong \mathbf{L}_{G \times K, V \oplus W}(A \times B).$$

In particular, for $A = B = *$, we obtain

$$\mathbf{L}_{G,V} \boxtimes \mathbf{L}_{K,W} \cong \mathbf{L}_{G \times K, V \oplus W},$$

i.e., the class $\{\mathbf{L}_{G,V}\}$, indexed by all compact Lie groups G and finite-dimensional G -representations V , is closed under forming box products up to natural isomorphism.

Proposition 3.15 ([13], Sect. I.3) The category \mathcal{IU} is closed symmetric monoidal under the box product, with monoidal unit the constant one-point orthogonal space $\mathbb{1}$. The right adjoint of $(-) \boxtimes Y: \mathcal{IU} \rightarrow \mathcal{IU}$ is a functor $\text{Hom}(Y, -)$ with value

$$\text{Hom}(X, Y)(V) = \mathcal{IU}(X, Y(- \oplus V))$$

on an inner product space $V \in \mathcal{IU}$.

Proposition 3.16 The functor $(-)(\mathbb{R}^\infty): \mathcal{IU} \rightarrow \mathcal{LU}$ is strong symmetric monoidal.

Recall that the target category \mathcal{LU} is only *weak* symmetric monoidal, though.

Proof. This follows from [13, Prop. I.7.24]. We omit the comparison of the associativity and commutativity transformations. The unit object $\mathbb{1} \in \mathcal{IU}$ is obviously sent to the singleton $* \in \mathcal{LU}$. \square

Corollary 3.17 The functor $(-)(\mathbb{R}^\infty): \mathcal{IU} \rightarrow \mathcal{LU}$ takes values in $*$ -modules.

Proof. Let $Y \in \mathcal{IU}$. The map $\lambda: Y(\mathbb{R}^\infty) \boxtimes_{\mathcal{L}} * \rightarrow Y(\mathbb{R}^\infty)$ agrees with the chain of natural isomorphisms

$$Y(\mathbb{R}^\infty) \boxtimes_{\mathcal{L}} * \cong (Y \boxtimes \mathbb{1})(\mathbb{R}^\infty) \cong Y(\mathbb{R}^\infty).$$

\square

In particular, the spaces $\mathbf{L}_{G,V}(\mathbb{R}^\infty) = \mathbf{L}(V, \mathbb{R}^\infty)/G$ are $*$ -modules.

Corollary 3.18 The functor $(-)(\mathbb{R}^\infty): \mathcal{IU} \rightarrow \mathcal{M}_*$ is strong symmetric monoidal.

3.3. Model categories of orthogonal spaces

In this section, we introduce a notion of *global equivalence* of orthogonal spaces, establish a corresponding *global model structure* and examine its relationship to \mathcal{L} -spaces.

A notion of equivalence that is meant to be a “global” one should yield some kind of G -equivariant equivalence after restricting the orthogonal spaces at hand to inner product spaces which are G -representations. So one could define a global equivalence $X \rightarrow Y$ to be a natural transformation that induces G -weak equivalences on homotopy colimits with respect to all finite-dimensional G -representations and G -equivariant linear isometric embeddings for all compact Lie groups G . This idea is the motivation for the more elementary definition given in [13, Rem. I.1.3], which is as follows.

Definition 3.19 A morphism $f: X \rightarrow Y$ of orthogonal spaces is a *global equivalence* if for all compact Lie groups G , all finite-dimensional G -representations V , all $k \geq 0$ and all commuting squares

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{\alpha} & X(V)^G \\ \text{incl} \downarrow & & \downarrow f(V)^G \\ D^k & \xrightarrow{\beta} & Y(V)^G \end{array}$$

there is a finite-dimensional G -representation W , a G -equivariant linear isometric embedding $\varphi: V \rightarrow W$ and a map $\lambda: D^k \rightarrow X(W)^G$ such that in the extended diagram

$$\begin{array}{ccccc} S^{k-1} & \xrightarrow{\alpha} & X(V)^G & \xrightarrow{X(\varphi)^G} & X(W)^G \\ \text{incl} \downarrow & & & \nearrow \lambda & \downarrow f(W)^G \\ D^k & \xrightarrow{\beta} & Y(V)^G & \xrightarrow{Y(\varphi)^G} & Y(W)^G \end{array}$$

the upper triangle commutes on the nose and the lower triangle commutes up to homotopy relative to S^{k-1} .

Example 3.20 The morphism $\Lambda_{X,Y}: X \boxtimes Y \rightarrow X \times Y$, see Example 3.13, is a global equivalence for all $X, Y \in \mathcal{I}\mathcal{U}$, as proven in [13, Thm. I.3.7].

In order to find a “global” model structure for orthogonal spaces, Schwede first establishes the *strong level model structure* in which weak equivalences and fibrations can be detected levelwise (i.e. “dimensionwise”) in terms of the standard model structures for $O(m)$ -spaces, cf. Example A.35. The *global model structure* is then obtained via a left Bousfield localization process. We give a brief summary of this approach.

Definition 3.21 A morphism $f: X \rightarrow Y$ of orthogonal spaces is a *strong level equivalence* (respectively *fibration*) if the induced maps on G -fixed points $X(V)^G \rightarrow Y(V)^G$ are weak homotopy equivalences (respectively Serre fibrations) for all compact Lie groups G and all finite-dimensional G -representations V .

Equivalently, $f(\mathbb{R}^m)$ is a weak equivalence (respectively fibration) of $O(m)$ -spaces for all $m \geq 0$, see [13, Lemma I.4.17]. In general, any choice of a family of model structures

for $O(m)$ -spaces, $m \geq 0$, yields a levelwise defined model structure for \mathcal{U} , provided that a certain compatibility condition is satisfied. We only present the *strong level model structure* as a special case and refer to [13, Sect. I.4] for a more general treatment.

Proposition 3.22 (Strong level model structure for \mathcal{U}) The category \mathcal{U} of orthogonal spaces admits a cofibrantly generated model structure with weak equivalences and fibrations the strong level equivalences and fibrations, respectively. It is proper, topological and symmetric monoidal. A set of generating cofibrations is given by

$$I^{\text{str}} = \{\mathbf{L}_{H, \mathbb{R}^m} \times i_k^{\mathbb{Q}} \mid m \geq 0, H \leq O(m)\}$$

for all closed subgroups $H \leq O(m)$. A set of generating acyclic cofibrations can be defined analogously in terms of the maps $j_k^{\mathbb{Q}}$.

Proof. This is proven in [13, I.4.22]. □

The cofibrations in the strong level model structure are sometimes called *flat cofibrations*. They can be described explicitly in terms of suitable skeleton filtrations of orthogonal spaces. We refer to [13, Sect. I.4] for details.

A (left) Bousfield localization process applied to the strong level model structure yields the desired *global model structure*. As usual, the fibrations in the localized model category must satisfy an extra assumption.

Definition 3.23 A morphism $f: X \rightarrow Y$ is a *global fibration* if it is a strong level fibration and the diagram

$$\begin{array}{ccc} X(V)^G & \xrightarrow{X(\phi)^G} & X(W)^G \\ f(V)^G \downarrow & & \downarrow f(W)^G \\ Y(V)^G & \xrightarrow{Y(\phi)^G} & Y(W)^G \end{array}$$

is homotopy cartesian (in the Quillen model structure for spaces) for all compact Lie groups G and all G -equivariant linear isometric embeddings $\phi: V \rightarrow W$ between faithful G -representations of finite dimension.

Theorem 3.24 (Global model structure for \mathcal{U} , cf. [13], Sect. I.5) The global equivalences and global fibrations are the weak equivalences and fibrations, respectively, in a proper, topological, cofibrantly generated model structure on \mathcal{U} .

As the class of cofibrations does not change under a left localization of model structures, I^{str} is also a set of generating cofibrations for the global model structure, and the notion of a cofibrant orthogonal space is unambiguous.

Proposition 3.25 The category of orthogonal spaces is a symmetric monoidal model category with respect to the global model structure and the box product.

Proof. The pushout product axiom is verified in [13, Prop. I.5.17]. \square

The categories of \mathcal{L} -spaces and orthogonal spaces can be compared via the adjunction

$$\mathcal{IU} \begin{array}{c} \xrightarrow{(-)(\mathbb{R}^\infty)} \\ \xleftarrow{u} \end{array} \mathcal{LU}.$$

We investigate the adjoint functors from the model category point of view.

Definition 3.26 (cf. [13], I.7.18) An \mathcal{L} -space X is called *injective* if the inclusion of fixed point sets $X^{\mathcal{L}[G,V]} \rightarrow X^G$ is a weak homotopy equivalence for all \mathfrak{C} -subgroups $G \leq \mathcal{L}(1)$ and all faithful finite-dimensional G -representations V .

Lemma 3.27 Let $Y \in \mathcal{IU}$ be cofibrant and let $X \in \mathcal{LU}$ be an injective \mathcal{L} -space. Then a morphism $Y(\mathbb{R}^\infty) \rightarrow X$ is a global equivalence of \mathcal{L} -spaces if and only if its adjoint $Y \rightarrow uX$ is a global equivalence of orthogonal spaces.

Proof. Both Y and uX send all morphisms in \mathcal{I} to closed embeddings in \mathcal{U} , as follows from the definition of uX , and from [13, I.4.21] in the case of Y . By [13, I.1.14], the map $f: Y \rightarrow uX$ is a global equivalence in \mathcal{IU} if and only if $f(\mathbb{R}^\infty)$ is a global equivalence in \mathcal{LU} . It suffices to show that the adjunction counit $\eta: (uX)(\mathbb{R}^\infty) \rightarrow X$ is a global equivalence if X is injective:

For each \mathfrak{C} -subgroup G , choose a cofinal sequence $V_1 \rightarrow V_2 \rightarrow \dots$ of faithful finite-dimensional G -subrepresentations of \mathbb{R}_G^∞ . Then the underlying G -map of η is the unique map that corresponds to the compatible system of G -maps

$$\begin{array}{ccccccc} \cdots & \longrightarrow & (uX)(V_i) & \longrightarrow & (uX)(V_{i+1}) & \longrightarrow & \cdots \\ & & & \searrow & \downarrow & & \\ & & & & X & & \end{array}$$

whose induced maps on H -fixed points for a closed subgroup $H \leq G$ can be identified with the inclusions

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X^{\mathcal{L}[H, V_i]} & \longrightarrow & X^{\mathcal{L}[H, V_{i+1}]} & \longrightarrow & \cdots \\ & & & \searrow & \downarrow & & \\ & & & & X^H & & \end{array}$$

All maps in the latter diagram are weak homotopy equivalences since X is injective. Hence, η is a weak equivalence of G -spaces for all G . In particular, each η^G is a weak homotopy equivalence. \square

The lemma justifies that we defined global equivalences to be the \mathcal{C} -equivalences. Now it is conceivable to expect to find a global model structure for \mathcal{LU} such that the adjunction becomes a Quillen equivalence. Unfortunately, the two projective model structures for \mathcal{LU} fail to comply with this idea. The problem is that all \mathcal{L} -spaces, not only the injective ones, are fibrant, and that the class of cofibrations is “too small”.

Example 3.28 Recall that $\mathbf{L}_{G, V} \times S^{-1} \rightarrow \mathbf{L}_{G, V} \times D^0$ is a generating cofibration in the global model structure for \mathcal{IU} . Under the functor $(-)(\mathbb{R}^\infty)$, it is sent to the \mathcal{L} -map $\emptyset \rightarrow \mathbf{L}(V, \mathbb{R}^\infty)/G$. It is likely that neither in the \mathcal{C} - nor in the \mathfrak{F} -projective model structure, the latter \mathcal{L} -space is ever cofibrant, and we have already seen an example for this failure in 2.61. Thus, $(-)(\mathbb{R}^\infty)$ is not a left Quillen functor.

We will see in Chapter 4 that it is possible to construct improved model structures for \mathcal{LU} which are Quillen equivalent to the global model structure for \mathcal{IU} . As the functor $(-)(\mathbb{R}^\infty)$ commutes with the box products, we will also ask whether there are weak symmetric monoidal Quillen equivalences, and give a positive answer in Proposition 5.17.

3.4. Orbispaces and Elmendorf’s theorem

Let G be a compact Lie group. The usual homotopy theory of G -spaces (with respect to the standard model structure, see Example A.35) can be described equivalently in terms of O_G -spaces: These are space-valued functors defined on an *orbit category* O_G

whose objects are the closed subgroups of G . This result is known as *Elmendorf's Theorem* and can be found in [4]. In modern language, it establishes a Quillen equivalence between GU and a suitable model category of O_G -spaces.

The standard model structure for GU is just a special case of the \mathcal{C} -projective model structures for MU , where M is a topological monoid, see Theorem A.30. Similarly, Elmendorf's Theorem generalizes to the world of monoid actions. We sketch how it comes into play in global homotopy theory, concentrating on the \mathcal{C} -projective model structure. The obvious analogues with respect to the \mathfrak{F} -projective model structure are true; the general approach for monoids can be found in [13, Sect. A.1.16].

Definition 3.29 The *global orbit category* O_{gl} has objects the \mathcal{C} -subgroups of $\mathcal{L}(1)$. For two objects G and K , the space of morphisms $G \rightarrow K$ is given by

$$O_{\text{gl}}(G, K) = \mathcal{L}\mathcal{U}(\mathcal{L}/G, \mathcal{L}/K) \cong (\mathcal{L}/K)^G,$$

hence O_{gl} is a topological category.

Definition 3.30 An *orbispace* is a continuous contravariant functor $O_{\text{gl}} \rightarrow \mathcal{U}$. We write $O_{\text{gl}}\mathcal{U}$ for the category of orbispaces and natural transformations.

Remark 3.31 As always for categories $\mathcal{D}\mathcal{U}$ of diagram spaces $\mathcal{D} \rightarrow \mathcal{U}$, we write $O_{\text{gl}}\mathcal{U}$ for the category of orbispaces, though in this case, $O_{\text{gl}}^{op}\mathcal{U}$ might be even more appropriate. The notation in [13] is *orbispc*.

The categories $\mathcal{L}\mathcal{U}$ and $O_{\text{gl}}\mathcal{U}$ are related by an adjunction.

Definition 3.32 The *fixed point functor* $\Phi: \mathcal{L}\mathcal{U} \rightarrow O_{\text{gl}}\mathcal{U}$ sends an \mathcal{L} -space X to the orbispace $\Phi(X): G \rightarrow X^G$. An \mathcal{L} -map $f: X \rightarrow Y$ induces a map of morphism spaces $f_*: \mathcal{L}\mathcal{U}(\mathcal{L}/G, X) \rightarrow \mathcal{L}\mathcal{U}(\mathcal{L}/G, Y)$ for each $G \in O_{\text{gl}}$, hence gives rise to a morphism $\Phi(f): \Phi(X) \rightarrow \Phi(Y)$ under the natural isomorphism $(-)^G \cong \mathcal{L}\mathcal{U}(\mathcal{L}/G, -)$ from Corollary A.18.

Proposition 3.33 (cf. [13], Sect. A.1) The functor $\Phi: \mathcal{L}\mathcal{U} \rightarrow O_{\text{gl}}\mathcal{U}$ has a left adjoint $\Lambda: O_{\text{gl}}\mathcal{U} \rightarrow \mathcal{L}\mathcal{U}$ whose value at $Y \in O_{\text{gl}}\mathcal{U}$ is given by the choice of a coend

$$\int^{G \in O_{\text{gl}}} \mathcal{L}/G \times Y(G).$$

The *projective model structure* on $O_{\text{gl}}\mathcal{U}$ is defined in a way such that this adjunction becomes a Quillen adjunction with respect to the \mathcal{C} -projective model structure 2.59

on $\mathcal{L}\mathcal{U}$, i.e., such that the (right adjoint) functor to orbispaces detects and reflects fibrations and weak equivalences. The global analogue of Elmendorf's theorem then says that it is indeed a Quillen equivalence.

Theorem 3.34 (Projective model structure for $O_{\text{gl}}\mathcal{U}$; see [11], Thm. VI.5.2) The category $O_{\text{gl}}\mathcal{U}$ of orbispaces admits a model structure with weak equivalences (respectively fibrations) the morphisms that are weak homotopy equivalences (respectively Serre fibrations) at each object $G \in O_{\text{gl}}$.

Theorem 3.35 (Elmendorf's Theorem) The adjunction unit $Y \rightarrow \Phi\Lambda(Y)$ is an isomorphism if $Y \in O_{\text{gl}}\mathcal{U}$ is cofibrant in the projective model structure.

Proof. This is proven for arbitrary monoids in [13, Prop. A.1.16]. \square

Proposition 3.36 (cf. [13], Thm. I.7.33) The \mathcal{C} -projective global model category of \mathcal{L} -spaces is Quillen equivalent to the projective model category of orbispaces via the adjunction

$$(\mathcal{L}\mathcal{U})_{\mathcal{C}} \begin{array}{c} \xleftarrow{\Lambda} \\ \xrightarrow{\Phi} \end{array} (O_{\text{gl}}\mathcal{U})_{\text{proj}}$$

Proof. Inspection of the two model structures involved shows that the fixed point functor Φ is a right Quillen functor: Indeed, it even preserves and reflects all fibrations and weak equivalences. All objects $X \in (\mathcal{L}\mathcal{U})_{\text{proj}}$ are fibrant. Let $Y \in O_{\text{gl}}\mathcal{U}$ be a cofibrant object. Then an \mathcal{L} -map $\Lambda(Y) \rightarrow X$ is a global equivalence if and only if its adjoint $Y \rightarrow \Phi(X)$ is a weak equivalence by Elmendorf's Theorem 3.35. \square

Remark 3.37 As already indicated, there is a similar Quillen equivalence

$$(\mathcal{L}\mathcal{U})_{\mathfrak{F}} \begin{array}{c} \xleftarrow{\Lambda'} \\ \xrightarrow{\Phi'} \end{array} (O_{\text{gl}}^{\mathfrak{F}}\mathcal{U})_{\text{proj}}$$

in the case of faithful complete subgroups of $\mathcal{L}(1)$.

4. Localized model structures for \mathcal{L} -spaces

We have seen above in Section 3.3 that there is a symmetric monoidal model category $(\mathcal{I}\mathcal{U})_{\text{global}}$ of orthogonal spaces with weak equivalences the global equivalences, and that there are compatible global equivalences of \mathcal{L} -spaces, namely the \mathfrak{C} -equivalences. However, the functor $(-)(\mathbb{R}^\infty): \mathcal{I}\mathcal{U} \rightarrow \mathcal{L}\mathcal{U}$ cannot be a left Quillen functor with respect to the \mathfrak{C} -projective model structure for $\mathcal{L}\mathcal{U}$, see Example 3.28. Thus, we attempt to find enhanced model structures for \mathcal{L} -spaces that do not suffer from this inconvenience.

In [13, Sect. I.7], a *flat global model structure* $(\mathcal{L}\mathcal{U})_{\flat}$ is established on the category of \mathcal{L} -spaces such that there is a chain of Quillen equivalences

$$(\mathcal{I}\mathcal{U})_{\text{global}} \begin{array}{c} \xrightarrow{(-)(\mathbb{R}^\infty)} \\ \xleftarrow{u} \end{array} (\mathcal{L}\mathcal{U})_{\flat} \begin{array}{c} \xleftarrow{\text{id}} \\ \xrightarrow{\text{id}} \end{array} (\mathcal{L}\mathcal{U})_{\mathfrak{C}} \begin{array}{c} \xleftarrow{\Lambda} \\ \xrightarrow{\Phi} \end{array} (O_{\text{gl}}\mathcal{U})_{\text{proj}}.$$

The flat model structure is still not perfect as the pushout product axiom fails for the same reasons as it does in $(\mathcal{L}\mathcal{U})_{\mathfrak{C}}$. Our ultimate aim will thus be the construction of an improved *sharp model structure* $(\mathcal{L}\mathcal{U})_{\sharp}$ that is at the same time weakly symmetric monoidal as a model category and still fits into a chain of Quillen equivalences

$$(\mathcal{I}\mathcal{U})_{\text{global}} \begin{array}{c} \xrightarrow{(-)(\mathbb{R}^\infty)} \\ \xleftarrow{u} \end{array} (\mathcal{L}\mathcal{U})_{\sharp} \begin{array}{c} \xleftarrow{\text{id}} \\ \xrightarrow{\text{id}} \end{array} (\mathcal{L}\mathcal{U})_{\mathfrak{C}} \begin{array}{c} \xleftarrow{\Lambda} \\ \xrightarrow{\Phi} \end{array} (O_{\text{gl}}\mathcal{U})_{\text{proj}}.$$

In both cases, the idea is to use Theorem A.30 and its generalized version A.34 again to produce an intermediate model structure for $\mathcal{L}\mathcal{U}$ where the set of generating cofibrations is chosen with care to serve our purposes. Then a left Bousfield localization yields a new model structure that has the same well-behaved cofibrations, but now the weak equivalences will be the global equivalences. In order to transport the flat and sharp model structures for $\mathcal{L}\mathcal{U}$ to Quillen-equivalent structures for \mathcal{M}_* via our main theorem, Theorem 5.3, we must ensure that the localized model categories are still cofibrantly generated. This is taken care of by Hirschhorn's abstract machinery from [6], which we briefly introduce in the first section. The next two sections are dedicated to an exposition of the flat and sharp model structures, respectively.

4.1. Localizations of model categories

We introduce Hirschhorn's definition of a *cellular model category*, recall some of its ingredients, and present an existence theorem for localizations of cellular model categories that applies to the model structures of Theorem A.34. Our exposition follows [6] closely.

Definition 4.1 (cf. [6], 12.1.1) Let \mathcal{C} be a cofibrantly generated model category with generating sets of cofibrations and acyclic cofibrations I and J , respectively. Then it is a *cellular model category* if the following conditions hold:

- 1) The domains and codomains of morphisms in I are compact.
- 2) The domains of morphisms in J are small relative to the class I -cell.
- 3) All cofibrations are effective monomorphisms.

We recall some of the definitions involved.

Definition 4.2 (cf. [6], 10.9.1) A morphism $f: A \rightarrow B$ in a category \mathcal{C} that admits all pushouts is an *effective monomorphism* (sometimes called *regular monomorphism*) if

$$A \xrightarrow{f} B \rightrightarrows B \cup_A B$$

is an equalizer diagram in \mathcal{C} , where the two parallel arrows are the canonical maps to the pushout.

Lemma 4.3 (see [16], Thm. 3.1) A map in \mathcal{U} is an effective monomorphism if and only if it is a closed embedding.

In order to give a definition of compactness, we need to be able to consider subcomplexes of relative I -cell complexes. But the property of being a subcomplex of $X \rightarrow Y$ might depend on the choice of the cell decomposition which is not part of the data of a relative cell complex $X \rightarrow Y$. Thus, we define a *presented I -cell complex* $f: X \rightarrow Y$ [6, Def. 10.6.2] to be a pair consisting of a λ -sequence $X = X_0 \rightarrow X_1 \rightarrow \dots$ (where λ is an ordinal) such that f is isomorphic to the composition of the X_β , and the cell decomposition $\{(T^\beta, e^\beta, h^\beta)\}_{\beta < \lambda}$ such that

- a) T^β is a set, the *set of β -cells*,
- b) $e^\beta: T^\beta \rightarrow I$ is a map of sets, and

c) for $i \in T^\beta$, $e_i^\beta = (C_i \rightarrow D_i) \in I$ and morphisms $h_i^\beta: C_i \rightarrow X_\beta$, there are pushout diagrams

$$\begin{array}{ccc} \coprod_{T^\beta} C_i & \xrightarrow{\coprod h_i^\beta} & X_\beta \\ \downarrow & & \downarrow \\ \coprod_{T^\beta} D_i & \longrightarrow & X_{\beta+1}. \end{array}$$

We call $\coprod_{\beta < \lambda} T^\beta$ the *set of cells* of f . Its cardinality is called the *size* of f .

Now it makes sense to talk about subcomplexes of a presented (relative) cell complex.

Definition 4.4 Let $f: X \rightarrow Y$ be a presented relative I -cell complex as above, then a *subcomplex* of f is a presented relative I -cell complex $\tilde{f}: X \rightarrow \tilde{Y}$ such that \tilde{f} is isomorphic to the composition of a λ -sequence $X = \tilde{X}_0 \rightarrow \tilde{X}_1 \rightarrow \dots$ and the cell decomposition $\{(\tilde{T}^\beta, \tilde{e}^\beta, \tilde{h}^\beta)\}_{\beta < \lambda}$ satisfies the following conditions:

a) For all $\beta < \lambda$, $\tilde{T}^\beta \subseteq T^\beta$ is a subset such that $\tilde{e}^\beta = e^\beta|_{\tilde{T}^\beta}$.

b) There is a map of λ -sequences

$$\begin{array}{ccccccc} X & \xlongequal{\quad} & \tilde{X}_0 & \longrightarrow & \tilde{X}_1 & \longrightarrow & \tilde{X}_2 & \longrightarrow & \dots \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \\ X & \xlongequal{\quad} & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \dots \end{array}$$

such that the triangle below commutes for all $\beta < \lambda$, $i \in \tilde{T}^\beta$.

$$\begin{array}{ccc} C_i & \xrightarrow{\tilde{h}_i^\beta} & \tilde{X}_\beta \\ & \searrow h_i^\beta & \downarrow \\ & & X_\beta \end{array}$$

Definition 4.5 (cf. [6], 10.8.1) Let I be a set of morphisms in a cocomplete category \mathcal{C} and fix a cardinal κ . An object $A \in \mathcal{C}$ is κ -*compact relative to* I if, for every presented relative I -cell complex $f: X \rightarrow Y$, every morphism $A \rightarrow Y$ factors through a subcomplex of f of size at most κ . As usual, we call A *compact relative to* I if it is κ -compact for some cardinal κ .

Example 4.6 (cf. [6], 10.8.3) Let $\mathcal{C} = \mathcal{U}$ be the category of spaces, and consider the set $I = \{S^{k-1} \rightarrow D^k\}$ of generating cofibrations in the Quillen model structure. Then every finite cell complex is $|\mathbb{N}|$ -compact relative to I . A cell complex of size an infinite cardinal κ is κ -compact relative to I .

We also recall the definition of local equivalences.

Definition 4.7 (cf. [6], 3.1.4) Let \mathcal{C} be a model category and S a class of morphisms. An object $W \in \mathcal{C}$ is *S-local* if it is fibrant and for all morphisms $f: A \rightarrow B$ in S , the induced map $f^*: \mathcal{C}(B, W) \rightarrow \mathcal{C}(A, W)$ is a weak equivalence of homotopy function complexes. A morphism $g: X \rightarrow Y$ is an *S-local equivalence* if for all *S-local* objects W , the induced map $g^*: \mathcal{C}(Y, W) \rightarrow \mathcal{C}(X, W)$ is a weak equivalence of homotopy function complexes.

A discussion of homotopy function complexes can be found in [6, Ch. 17]. We only need to know that if \mathcal{C} is a topological model category, A and B are cofibrant objects and W is fibrant, then the map f^* is a weak equivalence of homotopy function complexes if and only if it is a weak homotopy equivalence of spaces.

It is clear that each morphism in S is itself an *S-local equivalence*. The original class of weak equivalences is also contained in the *S-local equivalences*.

Proposition 4.8 (cf. [6], 3.1.5) Every weak equivalence in \mathcal{C} is an *S-local equivalence*.

We can now state Hirschhorn's theorem on the existence of localizations of cellular model categories.

Theorem 4.9 (cf. [6], 4.1.1) Let \mathcal{C} be a left proper cellular model category and let S be a set of morphisms. Then \mathcal{C} admits a left proper, cellular model structure $L_S\mathcal{C}$ which is called the *left localization* at S . Its weak equivalences are the *S-local equivalences*, its cofibrations are the same cofibrations as before. The fibrant objects in $L_S\mathcal{C}$ are the *S-local objects*.

In particular, the localized model structure is again cofibrantly generated. This aspect is our main motivation to use the language of cellular model categories.

The localized model structure comes with fibrant approximations for all objects $X \in \mathcal{C}$. These can be constructed in a functorial way using the small object argument, hence there is an *S-local approximation functor* $Q: \mathcal{C} \rightarrow \mathcal{C}$ which can be used to detect *S-local equivalences*.

Lemma 4.10 Let $Q: \mathcal{C} \rightarrow \mathcal{C}$ be an S -local approximation functor and let X and Y be fibrant objects in the original model structure on \mathcal{C} . A morphism $f: X \rightarrow Y$ is an S -equivalence if and only if Qf is a weak equivalence in the original model structure.

Proof. Clearly, f is an S -equivalence if and only if Qf is. The cofibrant objects in the original and in the localized model structure are the same; all S -local objects are fibrant in the original model structure. By Whitehead's Theorem, see [12, Thm. 14.4.8], Qf is a weak equivalence if and only if the induced map $(Qf)_*: [C, QX] \rightarrow [C, QY]$ between sets of homotopy classes is a bijection for all cofibrant $C \in \mathcal{C}$. The homotopy relation on $\mathcal{L}\mathcal{U}(C, D)$ is defined in terms of cylinder objects, and each choice of a cylinder object for C with respect to the original model structure is a cylinder object for C with respect to the localized model structure by Proposition 4.8. Thus, the above sets of homotopy classes agree with those defined with respect to the localization. Whitehead's Theorem for the localized model structure then shows that Qf is a weak equivalence if and only if it is an S -local equivalence. \square

Proposition 4.11 The \mathcal{O} -projective model structure from Theorem A.34 exhibits $\mathcal{L}\mathcal{U}$ as a cellular model category for every possible choice of the family \mathcal{O} .

Proof. The \mathcal{O} -projective model structure is cofibrantly generated with sets of generating (acyclic) cofibrations $I_{\mathcal{O}}$ (respectively $J_{\mathcal{O}}$). All relative $I_{\mathcal{O}}$ -cell complexes are closed embeddings of underlying spaces. Thus, all domains of morphisms in $J_{\mathcal{O}}$ are small relative to $I_{\mathcal{O}}$ -cell by Corollary A.20. All cofibrations are closed embeddings as they are retracts of relative $I_{\mathcal{O}}$ -cell complexes. They are effective monomorphisms in $\mathcal{L}\mathcal{U}$, because limits in $\mathcal{L}\mathcal{U}$ can be formed in \mathcal{U} where the maps are effective monomorphisms by Lemma 4.3. The remaining part of this section is dedicated to the proof that all (co-)domains of maps in $I_{\mathcal{O}}$ are compact relative to $I_{\mathcal{O}}$, which is then a consequence of Proposition 4.15. \square

Definition 4.12 Let I be a collection of closed embeddings and consider a relative I -cell complex $X \rightarrow Y$. If e is a β -cell in $Y - X$ which is attached via the pushout

$$\begin{array}{ccc} A & \xrightarrow{h} & X_{\beta-1} \\ \downarrow & & \downarrow \\ B & \xrightarrow{H} & X_{\beta-1} \cup_A B \end{array}$$

where $(A \rightarrow B) \in I$, we define the *interior* of e to be the open subspace $\text{int}(e) \subseteq X_{\beta}$ homeomorphic to $B - A$ via H .

As pushouts and (possibly transfinite) sequential colimits along closed embeddings of \mathcal{L} -spaces commute with generalized fixed points functors, see Definition A.25 and Proposition A.27, each such functor $G: \mathcal{L}\mathcal{U} \rightarrow \mathcal{U}$ converts relative I -cell complexes into relative I^G -cell complexes, where $I^G = \{G(i) \mid i \in I\}$.

Lemma 4.13 Let $G: \mathcal{L}\mathcal{U} \rightarrow \mathcal{U}$ be a generalized fixed points functor as defined in A.25, and let I be a collection of closed embeddings. For each $A \rightarrow B$ in I , there is a homeomorphism

$$G(B) - G(A) \cong G(B - A)$$

which is compatible with the embeddings to $G(B)$. Hence, for all cells e in a relative I -cell complex $X \rightarrow Y$, there is a homeomorphism $\text{int}(G(e)) \cong G(\text{int}(e))$ which is compatible with the embeddings to $G(Y)$. Here, the interior $\text{int}(G(e))$ is taken with respect to the relative I^G -cell complex $G(Y)$.

Proof. Observe that G takes the closed embedding $A \subseteq B$ to a closed embedding $G(A) \subseteq G(B)$. We may assume that all embeddings are actual inclusions of subspaces and show that $G(B) - G(A) = G(B - A)$ as subspaces of B .

First let $x \in G(B - A)$, then $x \in G(B)$. Moreover, x is contained in $B - A$. If x was in $G(A)$, it had to be in A , which is a contradiction. Thus, $G(B - A) \subseteq G(B) - G(A)$.

Now let $x \in G(B) - G(A) \subseteq G(B)$. Assume, x is in A , then the intersection property implies $x \in G(A)$, which is impossible. Hence, x is in $B - A$, and by the intersection property, it is in $G(B - A)$. \square

The following lemma will be useful in order to extend open subsets of the β -skeleton of a relative cell complex to higher skeleta.

Lemma 4.14 Let $A \rightarrow B$ be the inclusion of a closed subspace. Fix a finite collection of points $p_i \in B - A$ and an open subset U of A . Then there exists an open subset U' of B such that $U' \cap A = U$ and $p_i \notin U'$ for all i .

Proof. Let $V = A - U$ be the complement, then both V and $V' = \{p_i\}$ are closed in B . The open set $U' = B - (V \cup V')$ has the required properties. \square

Proposition 4.15 Let $A \in \mathcal{U}$ be a compact space (in the usual sense) and assume that $X \rightarrow Y$ is a relative $I_{\mathcal{O}}$ -cell complex. For $W \in \mathcal{O}$, the image of an \mathcal{L} -map $f: W \times A \rightarrow Y$ can intersect the interiors of only finitely many cells of $Y - X$. Thus, $W \times A$ is compact with respect to $I_{\mathcal{O}} - \text{cell}$.

The proof is analogous to [6, 10.7.4], we carry it out for the convenience of the reader.

Proof. Write $G = \mathcal{L}\mathcal{U}(W, -)$ for the generalized fixed points functor and recall that it converts relative $I_{\mathcal{O}}$ -cell complexes into relative $I_{\mathcal{O}}^G$ -cell complexes for

$$I_{\mathcal{O}}^G = \{G(Z \times i_k^{\mathcal{Q}}) \mid Z \times i_k^{\mathcal{Q}} \in I_{\mathcal{O}}\}.$$

The \mathcal{L} -map $f: W \times A \rightarrow Y$ corresponds to a map of spaces $f': A \rightarrow GX$ where $GX \rightarrow GY$ is a relative $I_{\mathcal{O}}^G$ -cell complex. If $f'(a)$ is contained in $\text{int}(Ge) \cong G(\text{int}(e))$, where e is an $I_{\mathcal{O}}$ -cell, then $f(w, a) = f'(a)(w) \in \text{int}(e)$ for all $w \in W$. So it is enough to show that the image of f' intersects only finitely many $I_{\mathcal{O}}^G$ -cells of GX . We assume that the statement is wrong and show that this contradicts the compactness of A .

Step 1: Write C for the image of f' which is compact in GX . Choose one point from the interior of each of the infinitely many cells whose interior intersects C to obtain a subset $P \subseteq GX - C$. We will show that P has no accumulation point in C . For each $c \in C$, we construct an open neighbourhood U of c in GX such that $U \cap P$ is either empty or equals $\{c\}$ if $c \in P$. Let e_c be the unique cell that contains c in its interior, and choose an open neighbourhood U_c of c in $\text{int}(e_c)$ that does not contain any points of P , except maybe c .

Step 2: We use the Lemma of Zorn to enlarge U_c to the desired set U . Assume that $GX \rightarrow GY$ is of size γ and that $e_c \in T^\alpha$. Define $Z = \{(\beta, V)\}$ where $\alpha \leq \beta \leq \gamma$ and $V \subseteq (GX)_\beta$ is an open subset such that $V \cap (GX)_\alpha = U_c$ and $V \cap P$ is empty or equals $\{c\}$. We give Z the preorder such that $(\beta_1, V_1) < (\beta_2, V_2)$ if $\beta_1 < \beta_2$ and $V_2 \cap (GX)_{\beta_1} = V_1$. An upper bound for a chain $(\beta_i, V_i)_i$ is given by $(\bigcup_i \beta_i, \bigcup_i V_i)$, thus Z has a maximal element (β_m, V_m) .

Step 3: It remains to show that $\beta_m = \gamma$, then $U = V_m$ is the desired set. If $\beta_m < \gamma$, we can extend V_m to the higher skeleton $(GX)_{\beta_m+1}$ as follows: Consider a cell $e_j \in T^{\beta_m}$ that is attached via the pushout

$$\begin{array}{ccc} G(Z_j \times S^{k_j-1}) & \xrightarrow{h_j} & (GX)_{\beta_m} \\ \downarrow & & \downarrow \\ G(Z_j \times D^{k_j}) & \longrightarrow & (GX)_{\beta_m} \cup_{G(Z_j \times S^{k_j-1})} G(Z_j \times D^{k_j}). \end{array}$$

The preimage $h_j^{-1}(V_m)$ is an open subset of $G(Z_j \times S^{k_j-1})$. By the previous lemma, it can be “thickened” to an open subset of $G(Z_j \times D^{k_j})$ that avoids $c \in P$ if necessary.

Let V' be the union of V_m with all thickenings with respect to all $e_j \in T^{\beta_m}$. Then V' is open in $(GX)_{\beta_m+1}$ and it is easy to see that $(\beta_m, V_m) < (\beta_m + 1, V')$, which is a contradiction. \square

4.2. The flat global model structure

As mentioned above, the functor $(-)(\mathbb{R}^\infty): (\mathcal{I}\mathcal{U})_{\text{global}} \rightarrow (\mathcal{L}\mathcal{U})_{\mathfrak{C}}$ is not a left Quillen functor: It does not take all cofibrations of orthogonal spaces to cofibrations in the \mathfrak{C} -projective model structure. We fix this problem by adding the images of all generating cofibrations for $(\mathcal{I}\mathcal{U})_{\text{global}}$ to our set $I_{\mathfrak{C}}$ of generating cofibrations, following the approach in [13, Sect. I.7].

As a first step, we recall that the \mathcal{L} -spaces

$$\mathbf{L}_{G,V}(\mathbb{R}^\infty) = \mathbf{L}(V, \mathbb{R}^\infty)/G \cong \mathcal{L}/\mathcal{L}[G, V]$$

are indeed of the form \mathcal{L}/M for a biclosed submonoid $M \leq \mathcal{L}(1)$. Now let \mathcal{C}^b be the collection of biclosed submonoids of $\mathcal{L}(1)$ that consists of all \mathfrak{C} -subgroups and all stabilizers $\mathcal{L}[G, V]$ where G is a \mathfrak{C} -subgroup and V is a faithful finite-dimensional subrepresentation of \mathbb{R}^∞ . Then Theorem A.30 yields the \mathcal{C}^b -projective model structure for \mathcal{L} -spaces which we denote by $(\mathcal{L}\mathcal{U})_{\mathcal{C}^b}$. By construction, the adjunction

$$(\mathcal{I}\mathcal{U})_{\text{global}} \begin{array}{c} \xrightarrow{(-)(\mathbb{R}^\infty)} \\ \xleftarrow{u} \end{array} (\mathcal{L}\mathcal{U})_{\mathcal{C}^b}$$

is a Quillen adjunction, hence the class of cofibrations suits our purpose.

The \mathcal{C}^b -projective model structure is cellular by Proposition 4.11, and we would like to localize such that the weak equivalences become the global equivalences. To achieve this, we have to find a set of morphisms S such that for all morphisms $f: X \rightarrow Y$ between S -local objects, the following are equivalent:

- a) f is a \mathcal{C}^b -equivalence
- b) f is a global equivalence

The restriction map $\mathcal{L}(1) = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty) \rightarrow \mathbf{L}(V, \mathbb{R}^\infty)$ descends to a map of \mathcal{L} -spaces $\rho_{G,V}: \mathcal{L}/G \rightarrow \mathcal{L}/\mathcal{L}[G, V]$ that induces maps of fixed-points $X^{\mathcal{L}[G,V]} \rightarrow X^G$ for all $X \in \mathcal{L}\mathcal{U}$ by Corollary A.18. Let S be the set of morphisms $\{\rho_{G,V}: \mathcal{L}/G \rightarrow \mathcal{L}/\mathcal{L}[G, V]\}$

for all \mathfrak{C} -subgroups $G \leq \mathcal{L}(1)$ and all faithful finite-dimensional G -subrepresentations $V \subseteq \mathbb{R}^\infty$. As \mathcal{L}/G and $\mathcal{L}/\mathcal{L}[G, V]$ are cofibrant in $(\mathcal{L}\mathcal{U})_{\mathfrak{C}^b}$, an object $W \in \mathcal{L}\mathcal{U}$ is S -local if and only if each map

$$(\rho_{G,V})^*: \mathcal{L}\mathcal{U}(\mathcal{L}/\mathcal{L}[G, V], W) \rightarrow \mathcal{L}\mathcal{U}(\mathcal{L}/G, W)$$

is a weak homotopy equivalence. But these maps are just the inclusions of fixed points $W^{\mathcal{L}[G, V]} \rightarrow W^G$, hence the S -local objects are precisely the injective \mathcal{L} -spaces (as introduced in Definition 3.26).

The left localization with respect to the set S exists due to Theorem 4.9. All objects in the \mathfrak{C}^b -model structure are fibrant, hence the S -local equivalences are precisely the global equivalences by Lemma 4.10.

Lemma 4.16 The localized model structure with respect to S is right proper.

Proof. The S -local fibrations are in particular global fibrations, so the statement is a consequence of the right properness of the \mathfrak{C} -projective model structure $(\mathcal{L}\mathcal{U})_{\mathfrak{C}}$. \square

The localized model structure with respect to S is left proper by Hirschhorn's theorem. It is topological due to Corollary A.39. Thus, we have proven the following theorem, cf. [13, Thm. I.7.20].

Theorem 4.17 (Flat global model structure for $\mathcal{L}\mathcal{U}$) There is a cellular model structure $(\mathcal{L}\mathcal{U})_b$ for \mathcal{L} -spaces with weak equivalences the global equivalences and cofibrations generated by the set

$$I_{\mathfrak{C}^b} = \{\mathcal{L}/M \times i_k^{\mathbb{Q}} \mid M \in \mathfrak{C}^b\}.$$

An object is fibrant if and only if it is injective. The model structure is topological and proper.

Corollary 4.18 There is a chain of Quillen equivalences

$$(\mathcal{I}\mathcal{U})_{\text{global}} \xrightleftharpoons[u]{(-)(\mathbb{R}^\infty)} (\mathcal{L}\mathcal{U})_b \xrightleftharpoons[\text{id}]{\text{id}} (\mathcal{L}\mathcal{U})_{\mathfrak{C}} \xrightleftharpoons[\Phi]{\Lambda} (O_{\text{gl}}\mathcal{U})_{\text{proj}}.$$

Proof. The two adjunctions on the left hand side are Quillen equivalences by construction and Lemma 3.27, the one on the right is due to Elmendorf's Theorem 3.35. \square

Unfortunately, the flat global model structure is not compatible with the monoidal structure because every cofibration in $(\mathcal{L}\mathcal{U})_{\mathfrak{E}}$ is a cofibration in $(\mathcal{L}\mathcal{U})_{\mathfrak{b}}$, and we have seen in Section 2.6 that these maps do not satisfy the pushout product axiom. We remedy this defect in the next section.

4.3. The sharp global model structure

We modify the flat global model structure in order to obtain a model category of \mathcal{L} -spaces which is weakly symmetric monoidal and still Quillen-equivalent to $(\mathcal{I}\mathcal{U})_{\text{global}}$ and $(\mathcal{L}\mathcal{U})_{\mathfrak{E}}$. The main difference is that we do not use the collection of \mathcal{L} -spaces \mathcal{L}/K and $\mathcal{L}/\mathcal{L}[G, V]$ to define the cofibrations but rather its closure under the box product. Let \mathcal{O}_{\sharp} be the set $\{\mathcal{L}/K\} \cup \{\mathcal{L}/\mathcal{L}[G, V]\} \cup \{\mathcal{L}/K \boxtimes_{\mathcal{L}} \mathcal{L}/\mathcal{L}[G, V]\}$ for all \mathfrak{F} -subgroups $K \leq \mathcal{L}(1)$, all \mathfrak{E} -subgroups $G \leq \mathcal{L}(1)$ and all finite-dimensional faithful G -subrepresentations of \mathbb{R}^{∞} .

Lemma 4.19 Up to isomorphism, the set \mathcal{O}_{\sharp} is closed under forming box products.

Proof. We have already seen in Example 2.17 and 3.14 that up to isomorphism, the sets $\{\mathcal{L}/K\}$ and $\{\mathbf{L}_{G, V}\}$ are closed under box product, hence so is the image $\{\mathcal{L}/\mathcal{L}[G, V]\}$ of the latter under the monoidal functor $(-)(\mathbb{R}^{\infty})$. The same is then true for \mathcal{O}_{\sharp} by associativity and commutativity of the box product. \square

The objects $\mathcal{L}/K \boxtimes_{\mathcal{L}} \mathcal{L}/\mathcal{L}[G, V]$ are unlikely to be of the form \mathcal{L}/M for a biclosed submonoid $M \leq \mathcal{L}(1)$, hence do not represent any fixed points functor. However, all we need to get a model structure with generating cofibrations the maps $W \times i_k^Q$ for $W \in \mathcal{O}_{\sharp}$ is that each W represents a *generalized* fixed points functor as defined in A.25.

Lemma 4.20 For each \mathcal{L} -space $W = \mathcal{L}/K \boxtimes_{\mathcal{L}} \mathcal{L}/\mathcal{L}[G, V]$ contained in \mathcal{O}_{\sharp} , the functor $\mathcal{L}\mathcal{U}(W, -): \mathcal{L}\mathcal{U} \rightarrow \mathcal{U}$ is a generalized fixed points functor.

Proof. It is well-known that $\mathcal{L}\mathcal{U}(W, -)$ preserves (closed) embeddings. We construct a natural closed embedding $\mathcal{L}\mathcal{U}(W, X) \rightarrow X$ as follows: There is a chain of \mathcal{L} -maps

$$\mathcal{L}(1) \rightarrow \mathcal{L}/\overline{K \times G} \cong \mathcal{L}/K \boxtimes_{\mathcal{L}} \mathcal{L}/G \rightarrow \mathcal{L}/K \boxtimes_{\mathcal{L}} \mathcal{L}/\mathcal{L}[G, V] = W$$

where $\overline{K \times G}$ is an \mathfrak{F} -subgroup isomorphic to $K \times G$ and the last map is $\mathcal{L}/K \boxtimes_{\mathcal{L}} \rho_{G, V}$. We obtain a chain of natural map of spaces

$$\mathcal{L}\mathcal{U}(W, X) \rightarrow \mathcal{L}\mathcal{U}(\mathcal{L}/\overline{K \times G}, X) \rightarrow \mathcal{L}\mathcal{U}(\mathcal{L}(1), X) \cong X.$$

The map in the middle is an inclusion of fixed points, hence a closed embedding. The first map is naturally isomorphic to the closed embedding $(F_{\boxtimes_{\mathcal{L}}}(\mathcal{L}/K, X))^{\mathcal{L}[G, V]} \rightarrow (F_{\boxtimes_{\mathcal{L}}}(\mathcal{L}/K, X))^G$ under the adjunctions 2.21 and A.18.

Now let $X_0 \rightarrow X_1 \rightarrow \dots$ be any λ -sequence of closed embeddings. Then there are natural homeomorphisms

$$\begin{aligned} \operatorname{colim}_{\beta < \lambda} \mathcal{L}\mathcal{U}(\mathcal{L}/K \boxtimes_{\mathcal{L}} \mathcal{L}/\mathcal{L}[G, V], X_{\beta}) &\cong \operatorname{colim}_{\beta < \lambda} \mathcal{L}\mathcal{U}(\mathcal{L}/K, F_{\boxtimes_{\mathcal{L}}}(\mathcal{L}/\mathcal{L}[G, V], X_{\beta})) \\ &\cong \mathcal{L}\mathcal{U}(\mathcal{L}/K, \operatorname{colim}_{\beta < \lambda} F_{\boxtimes_{\mathcal{L}}}(\mathcal{L}/\mathcal{L}[G, V], X_{\beta})) \cong \mathcal{L}\mathcal{U}(\mathcal{L}/K, F_{\boxtimes_{\mathcal{L}}}(\mathcal{L}/\mathcal{L}[G, V], \operatorname{colim}_{\beta < \lambda} X_{\beta})) \\ &\cong \mathcal{L}\mathcal{U}(\mathcal{L}/K \boxtimes_{\mathcal{L}} \mathcal{L}/\mathcal{L}[G, V], \operatorname{colim}_{\beta < \lambda} X_{\beta}) \end{aligned}$$

whose composite is the canonical map, hence $\mathcal{L}\mathcal{U}(W, -)$ preserves sequential colimits along closed embeddings. The second isomorphism comes from the fact that \mathcal{L}/K is finite relative to closed embeddings, see Example A.24. This is also true for $\mathcal{L}/\mathcal{L}[G, V]$, so the third isomorphism follows from Corollary 5.21.

Finally, we verify the intersection property. Let $f: X \rightarrow Y$ be an embedding of \mathcal{L} -spaces and consider the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{L}\mathcal{U}(W, X) & \longrightarrow & \mathcal{L}\mathcal{U}(\mathcal{L}/\overline{K \times G}, X) & \longrightarrow & X \\ \downarrow f_* & & \downarrow f_* & & \downarrow f \\ \mathcal{L}\mathcal{U}(W, Y) & \longrightarrow & \mathcal{L}\mathcal{U}(\mathcal{L}/\overline{K \times G}, Y) & \longrightarrow & Y \end{array}$$

The right hand square has the intersection property because $\mathcal{L}/\overline{K \times G}$ represents a fixed points functor. Thus, it suffices to show that the left hand square has the intersection property. This is true because it is naturally isomorphic to the square

$$\begin{array}{ccc} (F_{\boxtimes_{\mathcal{L}}}(\mathcal{L}/K, X))^{\mathcal{L}[G, V]} & \longrightarrow & (F_{\boxtimes_{\mathcal{L}}}(\mathcal{L}/K, X))^G \\ \downarrow f_* & & \downarrow f_* \\ (F_{\boxtimes_{\mathcal{L}}}(\mathcal{L}/K, Y))^{\mathcal{L}[G, V]} & \longrightarrow & (F_{\boxtimes_{\mathcal{L}}}(\mathcal{L}/K, Y))^G \end{array}$$

which has the intersection property. □

Thus, all objects in the set \mathcal{O}_{\sharp} represent generalized fixed points functors. By Theorem A.34, we obtain the \mathcal{O}_{\sharp} -projective model structure $(\mathcal{L}\mathcal{U})_{\mathcal{O}_{\sharp}}$. Its weak equivalences are the \mathcal{O}_{\sharp} -equivalences which are detected by the family of functors $\mathcal{L}\mathcal{U}(W, -): \mathcal{L}\mathcal{U} \rightarrow \mathcal{U}$ for $W \in \mathcal{O}_{\sharp}$. Sets of generating cofibrations $I_{\mathcal{O}_{\sharp}}$ and acyclic cofibrations $J_{\mathcal{O}_{\sharp}}$ are given by

$$I_{\mathcal{O}_{\sharp}} = \{W \times i_k^{\mathcal{Q}}\}, \quad J_{\mathcal{O}_{\sharp}} = \{W \times j_k^{\mathcal{Q}}\}.$$

Lemma 4.21 The \mathcal{O}_{\sharp} -projective model structure on $\mathcal{L}\mathcal{U}$ is weakly symmetric monoidal in the sense of Definition 2.46.

Proof. Let W' and W'' be elements of \mathcal{O}_{\sharp} , then $W' \boxtimes_{\mathcal{L}} W''$ is isomorphic to some object $W \in \mathcal{O}_{\sharp}$. The computation in 2.48 shows that the pushout product of generating cofibrations $W' \times i_k^{\mathcal{Q}}$ and $W'' \times i_m^{\mathcal{Q}}$ is isomorphic to $W \times i_{k+m}^{\mathcal{Q}}$. Similarly, we have $(W' \times i_k^{\mathcal{Q}}) \square (W'' \times j_m^{\mathcal{Q}}) \cong W \times (i_k^{\mathcal{Q}} \square j_m^{\mathcal{Q}})$ which is an \mathcal{L} -homotopy equivalence because $i_k^{\mathcal{Q}} \square j_m^{\mathcal{Q}}$ (where the pushout product is taken in \mathcal{U} with respect to the product) is a homotopy equivalence. Each \mathcal{L} -homotopy equivalence is an \mathcal{O}_{\sharp} -equivalence because $\mathcal{L}\mathcal{U}(W, -): \mathcal{L}\mathcal{U} \rightarrow \mathcal{U}$ is a continuous functor. \square

As in Section 4.2, we localize this intermediate model structure in a way such that the local equivalences are the global equivalences. We have to relate all elements of \mathcal{O}_{\sharp} by global equivalences. Recall that there are global equivalences (even strong \mathfrak{F} -equivalences)

$$\rho_{G,V}: \mathcal{L}/G = \mathbf{L}(\mathbb{R}_G^{\infty}, \mathbb{R}^{\infty})/G \rightarrow \mathbf{L}(V, \mathbb{R}^{\infty})/G = \mathcal{L}/\mathcal{L}[G, V].$$

These induce strong \mathfrak{F} -equivalences

$$\mathcal{L}/K \boxtimes_{\mathcal{L}} \rho_{G,V}: \mathcal{L}/K \boxtimes_{\mathcal{L}} \mathcal{L}/G \rightarrow \mathcal{L}/K \boxtimes_{\mathcal{L}} \mathcal{L}/\mathcal{L}[G, V]$$

where the domain is isomorphic to the \mathcal{L} -space $\mathcal{L}/\overline{K} \times \overline{G}$ for an \mathfrak{F} -subgroup $\overline{K} \times \overline{G}$. If $K \leq \mathcal{L}(1)$ is an \mathfrak{F} -subgroup and G is a \mathfrak{E} -subgroup such that both are isomorphic as Lie groups, then there is a linear isometric embedding $i_{K,G}: \mathbb{R}_K^{\infty} \rightarrow \mathbb{R}_G^{\infty}$ which is equivariant under this isomorphism. Note that the map $i_{K,G}$ is not canonical, but all choices are isomorphic and form a set. Each such choice of $i_{K,G}$ induces an \mathcal{L} -map

$$i_{K,G}^*: \mathcal{L}/G = \mathbf{L}(\mathbb{R}_G^{\infty}, \mathbb{R}^{\infty}) \rightarrow \mathbf{L}(\mathbb{R}_K^{\infty}, \mathbb{R}^{\infty})/K = \mathcal{L}/K$$

which is a strong \mathfrak{F} -equivalence by Corollary 2.54.

Define S_{\sharp} to be the set

$$S_{\sharp} = \{\rho_{G,V}\} \cup \{\mathcal{L}/K \boxtimes_{\mathcal{L}} \rho_{G,V}\} \cup \{i_{K,G}\}$$

of all these morphisms. The letters K and G always denote an \mathfrak{F} -subgroup and a \mathcal{C} -subgroup, respectively, and V is always a finite-dimensional faithful subrepresentation of \mathbb{R}_G^{∞} .

Theorem 4.22 (Sharp global model structure for $\mathcal{L}\mathcal{U}$) There is a cellular model structure $(\mathcal{L}\mathcal{U})_{\sharp}$ for \mathcal{L} -spaces with weak equivalences the global equivalences and cofibrations generated by the set

$$I_{\mathcal{O}_{\sharp}} = \{W \times i_k^{\mathcal{O}} \mid W \in \mathcal{O}_{\sharp}\}.$$

An object is fibrant if and only if it is S_{\sharp} -local. The model structure is topological, proper and weakly symmetric monoidal.

Proof. The model category $(\mathcal{L}\mathcal{U})_{\mathcal{O}_{\sharp}}$ is cellular by Proposition 4.11, thus the left localization with respect to S_{\sharp} exists. It is left proper and cellular with cofibrations and fibrant objects as described above. By construction, a map between S_{\sharp} -local objects is an S_{\sharp} -local equivalence if and only if it is a global equivalence. As all objects in $(\mathcal{L}\mathcal{U})_{\mathcal{O}_{\sharp}}$ are fibrant, Lemma 4.10 implies that the weak equivalences in $(\mathcal{L}\mathcal{U})_{\sharp}$ are precisely the global equivalences. Right properness is immediate because each fibration in the localized model structure is a fibration in the right proper model category $(\mathcal{L}\mathcal{U})_{\mathcal{C}}$. The sharp model structure is topological by Corollary A.39.

We verify that $(\mathcal{L}\mathcal{U})_{\sharp}$ is weakly symmetric monoidal: The first part of the pushout product axiom holds because it does in $(\mathcal{L}\mathcal{U})_{\mathcal{O}_{\sharp}}$ and the cofibrations have not changed. Now let $f: X \rightarrow Y$ and $f': X' \rightarrow Y'$ be any cofibrations such that f' is a global equivalence and consider the following commutative diagram.

$$\begin{array}{ccc} X \times X' & \xrightarrow{X \times f'} & X \times Y' \\ f \times X' \downarrow & \nearrow g & \swarrow \leftarrow \\ & P & \searrow \leftarrow \\ Y \times X' & \xrightarrow{Y \times f'} & Y \times Y' \end{array}$$

where P is the pushout. The two horizontal maps are global equivalences since f' is, and the map $f \times X'$ is an h -cofibration. The Gluing Lemma A.38 implies that g is

a global equivalence, hence so is the pushout product q . The remaining part of the pushout product axiom follows by symmetry. \square

The functor $(-)(\mathbb{R}^\infty): \mathcal{I}\mathcal{U} \rightarrow \mathcal{L}\mathcal{U}$ is strong symmetric monoidal, thus the same proof as for Corollary 4.18 yields the next comparison result.

Corollary 4.23 There is a chain of Quillen equivalences

$$(\mathcal{I}\mathcal{U})_{\text{global}} \begin{array}{c} \xrightarrow{(-)(\mathbb{R}^\infty)} \\ \xleftarrow{u} \end{array} (\mathcal{L}\mathcal{U})_{\sharp} \begin{array}{c} \xleftarrow{\text{id}} \\ \xrightarrow{\text{id}} \end{array} (\mathcal{L}\mathcal{U})_{\mathfrak{C}} \begin{array}{c} \xleftarrow{\Lambda} \\ \xrightarrow{\Phi} \end{array} (\mathcal{O}_{\text{gl}}\mathcal{U})_{\text{proj}}$$

where the adjoint functors on the left form a symmetric monoidal Quillen equivalence of (weak) symmetric monoidal model categories.

5. Model structures for $*$ -modules

In [1, Sect. 4], Blumberg, Cohen and Schlichtkrull established a non-equivariant model structure for $*$ -modules which is symmetric monoidal, see Proposition 2.49. Their proof relies on certain “lifting” results that show how to transport a model structure along the free algebra functor of a monad. In the present chapter, we recall their approach and generalize it. Our main theorem shows that every well-behaved model structure for \mathcal{LU} can be lifted to a Quillen equivalent model structure for \mathcal{M}_* . In particular, it applies to all global model structures described in Chapters 2 and 4. The sharp global model structure for \mathcal{LU} then lifts to a symmetric monoidal global model structure for $*$ -modules which is monoidally Quillen equivalent to the global model category of orthogonal spaces. All three model structures lift to the associated categories of monoids and modules, which are then Quillen equivalent by the induced pairs of adjoint functors.

We present our main result, the Transport Theorem, in Section 5.1. Then we concentrate on the non-equivariant situation first and recall the approach used in [1]. Some of its techniques will be crucial for the proof of the Transport Theorem given in Section 5.4. Many applications can be found in Section 5.3, including the sharp model structure for \mathcal{LU} . Finally, we discuss the categories of monoids and modules over \mathcal{IU} , \mathcal{LU} and \mathcal{M}_* .

5.1. Transporting model structures to $*$ -modules

In this section, we discuss our main result which transports well-behaved model structures for \mathcal{L} -spaces to Quillen equivalent model structures for $*$ -modules. We refer to it by the name “Transport Theorem”. It can be applied to most of the model structures for \mathcal{LU} described in Chapter 2 and Chapter 4, as presented in detail in Sections 5.2 and 5.3.

Definition 5.1 A class of morphisms in \mathcal{LU} is called *admissible* if it is stable under transfinite composition and contains the class of strong \mathfrak{F} -equivalences.

If such a class \mathcal{W} has the 2-out-of-3 property, then it is preserved by the functor $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{LU} \rightarrow \mathcal{LU}$ because the natural map $\bar{\lambda}: Y \rightarrow F_{\boxtimes_{\mathcal{L}}}(*, Y)$ is a strong \mathfrak{F} -equivalence. In particular, this holds for some interesting classes of morphisms identified by the next lemma.

Lemma 5.2 The following classes of weak equivalences \mathcal{W} of \mathcal{L} -spaces (as defined in Section 2.6) are admissible.

- i) The class of \mathfrak{F} -equivalences
- ii) The class of \mathfrak{C} -equivalences
- iii) The class of (underlying) homotopy equivalences
- iv) The class of (underlying) weak homotopy equivalences

Write \mathcal{W}_* for the subclass of morphisms in \mathcal{W} whose domains and codomains are $*$ -modules. Note that except for the class of homotopy equivalences, all choices of \mathcal{W} are \mathfrak{C} -equivalences in the sense of Definition A.29.

Theorem 5.3 (Transport Theorem) Let $(\mathcal{L}\mathcal{U})_a$ be any cofibrantly generated model structure on the category of \mathcal{L} -spaces with an admissible class of weak equivalences \mathcal{W} and sets I and J of generating cofibrations and acyclic cofibrations, respectively. Assume in addition that all maps in I and J are closed embeddings of underlying spaces. Then the category of $*$ -modules admits a cofibrantly generated model structure $(\mathcal{M}_*)_a$ with weak equivalences the class \mathcal{W}_* and fibrations detected by the functor $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{M}_* \rightarrow (\mathcal{L}\mathcal{U})_a$. Sets of generating cofibrations and generating acyclic cofibrations are given by $I_* = I \boxtimes_{\mathcal{L}} *$ and $J_* = J \boxtimes_{\mathcal{L}} *$, respectively. The adjunction

$$(\mathcal{L}\mathcal{U})_a \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}}*} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(*, -)} \end{array} (\mathcal{M}_*)_a$$

is a Quillen equivalence.

The Transport Theorem is proven in Section 5.4. In the following, we give sufficient conditions for the model structure $(\mathcal{M}_*)_a$ to be proper, topological, and symmetric monoidal.

Lemma 5.4 If $(\mathcal{L}\mathcal{U})_a$ is right proper, then so is $(\mathcal{M}_*)_a$. If \mathcal{W} is one of the classes from Lemma 5.2 and all elements of I are h -cofibrations in $\mathcal{L}\mathcal{U}$, then $(\mathcal{M}_*)_a$ is left proper.

Proof. Right properness is shown in Corollary 5.10. For left properness, consider the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \downarrow & & \downarrow \\ Z & \xrightarrow{g} & W \end{array}$$

in \mathcal{M}_* where f is in \mathcal{W}_* and i is a cofibration. The inclusion functor $l: \mathcal{M}_* \rightarrow \mathcal{LU}$ is a left adjoint, hence

$$\begin{array}{ccc} l(X) & \xrightarrow{l(f)} & l(Y) \\ l(i) \downarrow & & \downarrow \\ l(Z) & \xrightarrow{l(g)} & l(W) \end{array}$$

is a pushout in \mathcal{LU} and $l(f)$ is in \mathcal{W} . The map $l(i)$ might not be a cofibration in $(\mathcal{LU})_a$. However, all continuous functors preserve h -cofibrations, thus every map in $I \boxtimes_{\mathcal{L}} *$ and hence every cofibration in $(\mathcal{M}_*)_a$ is an h -cofibration. As l is continuous by definition of the topological enrichment of \mathcal{LU} , $l(i)$ is an h -cofibration and the Gluing Lemma A.38 implies that $l(g)$ is in \mathcal{W} . \square

Lemma 5.5 The model category $(\mathcal{M}_*)_a$ is topological if $(\mathcal{LU})_a$ is a topological model category.

Proof. Let $f: X \rightarrow Y$ be a generating cofibration for $(\mathcal{LU})_a$ and $i: A \rightarrow B$ any cofibration in \mathcal{U} . By assumption, the pushout product

$$f \square i: P = Y \times A \cup_{X \times A} X \times B \rightarrow Y \times B$$

is again a cofibration in $(\mathcal{LU})_a$. The map $f \boxtimes_{\mathcal{L}} *$ is a generating cofibration in $(\mathcal{M}_*)_a$ whose pushout product with i is isomorphic to

$$(f \square i) \boxtimes_{\mathcal{L}} *: P \boxtimes_{\mathcal{L}} * \rightarrow (Y \times B) \boxtimes_{\mathcal{L}} *.$$

As $- \boxtimes_{\mathcal{L}} *: (\mathcal{LU})_a \rightarrow (\mathcal{M}_*)_a$ is a left Quillen functor, this map is a cofibration in \mathcal{M}_* . If f is a generating acyclic cofibration or i any acyclic cofibration, then $f \square i$ is an acyclic cofibration in \mathcal{LU} , hence so is $(f \boxtimes_{\mathcal{L}} *) \square i \cong (f \square i) \boxtimes_{\mathcal{L}} *$ in \mathcal{M}_* . \square

Proposition 5.6 Assume in addition that $(\mathcal{L}\mathcal{U})_a$ is a weak symmetric monoidal model structure in the sense of Definition 2.46 with respect to the box product. Then $(\mathcal{M}_*)_a$ is a symmetric monoidal model category.

Proof. Note that there are natural isomorphisms

$$(X \boxtimes_{\mathcal{L}} *) \boxtimes_{\mathcal{L}} (X' \boxtimes_{\mathcal{L}} *) \cong (X \boxtimes_{\mathcal{L}} X') \boxtimes_{\mathcal{L}} *$$

for all \mathcal{L} -spaces X and X' . Similar reasoning as in the previous proof then shows that for two generating cofibrations $f: A \rightarrow B$ and $f': A' \rightarrow B'$ for $(\mathcal{L}\mathcal{U})_a$, the pushout product of $f \boxtimes_{\mathcal{L}} *$ and $f' \boxtimes_{\mathcal{L}} *$ is isomorphic to $(f \square f') \boxtimes_{\mathcal{L}} *$, hence is a cofibration in \mathcal{M}_* , and acyclic if f or f' is a generating acyclic cofibration. \square

5.2. The non-equivariant model structure

The first model structures for \mathcal{L} -spaces and $*$ -modules which have been studied are *non-equivariant* model structures in the sense that the weak equivalences and fibrations are defined after forgetting the $\mathcal{L}(1)$ -actions. We have already seen the model category $(\mathcal{L}\mathcal{U})_{\text{non}}$ in Prop. 2.44, but we describe yet another approach to obtain it, taking the monad point-of-view. A non-equivariant model structure for \mathcal{M}_* is obtained in the same way, cf. [1, Sect. 4.6], and this very simple setting will serve as a “toy example” for us in order to explain our strategy for the construction of model categories for $*$ -modules.

Recall that we can describe the categories of \mathcal{L} -spaces and co- $*$ -modules in terms of the two monads $\mathbb{L} = \mathcal{L}(1) \times -$ and $\mathbb{F} = F_{\boxtimes_{\mathcal{L}}}(*, -)$ via isomorphisms of categories $\mathcal{L}\mathcal{U} \cong \mathcal{U}[\mathbb{L}]$ and $\mathcal{M}^* \cong \mathcal{L}\mathcal{U}[\mathbb{F}]$ (see Section 2.5). We can arrange these identifications in the following diagram which displays the two adjunctions consisting of the free and forgetful functors of \mathbb{L} - (respectively \mathbb{F} -) algebras and the equivalence of categories between \mathcal{M}^* and \mathcal{M}_* given by the box product.

$$\mathcal{U} \begin{array}{c} \xrightarrow{\mathcal{L}(1) \boxtimes_{\mathcal{L}} -} \\ \xleftarrow{\text{forget}} \end{array} \mathcal{U}[\mathbb{L}] \cong \mathcal{L}\mathcal{U} \begin{array}{c} \xrightarrow{F_{\boxtimes_{\mathcal{L}}}(*, -)} \\ \xleftarrow{R} \end{array} \mathcal{L}\mathcal{U}[\mathbb{F}] \cong \mathcal{M}^* \begin{array}{c} \xrightarrow{- \boxtimes_{\mathcal{L}} *} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(*, -)} \end{array} \mathcal{M}_* \quad (5.7)$$

Now let T be a monad on the model category \mathcal{C} . There are various “lifting theorems” giving sufficient conditions for the model structure on \mathcal{C} to “lift” along the free T -algebra functor to the category of algebras $\mathcal{C}[T]$. Whenever we establish a model

structure for $*$ -modules, we will proceed as follows: We start with a nice model structure on \mathcal{U} or \mathcal{LU} , lift it along the free functor(s) to \mathcal{M}^* and then transport it to \mathcal{M}_* along the equivalence of categories. Note that this approach is due to Blumberg, Cohen and Schlichtkrull, see [1, Sect. 4.6], and relies on some techniques developed in [5].

The following lifting theorem perfectly suits our situation.

Theorem 5.8 (Lifting of model structures) Let \mathcal{C} be a cofibrantly generated model category and I (respectively J) a set of generating (acyclic) cofibrations. Let T be a monad on \mathcal{C} and denote by I_T and J_T the images of I and J , respectively, under the free T -algebra functor. Assume that

- (R1) the domains of I_T and J_T are small relative to I_T -cell and J_T -cell, respectively;
- (R2) every morphism in J_T -cell is sent to a weak equivalence in \mathcal{C} under the forgetful functor;
- (R3) the category $\mathcal{C}[T]$ of T -algebras is cocomplete.

Then $\mathcal{C}[T]$ is a cofibrantly generated model category with generating sets of (acyclic) cofibrations I_T (respectively J_T), and weak equivalences and fibrations detected by the forgetful functor to \mathcal{C} .

Proof. The statement of our lifting theorem agrees with the result [14, Lemma 2.3.] except that Schwede and Shipley require the monad to commute with filtered colimits in order to prove that $\mathcal{C}[T]$ is cocomplete. In our formulation, this is guaranteed by requirement (R3). \square

In our situation, requirement (R3) is always satisfied because the categories \mathcal{LU} and \mathcal{M}_* are bicomplete.

Corollary 5.9 In the setting of Theorem 5.8, the adjunction

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{R} \end{array} \mathcal{C}[T]$$

consisting of the free T -algebra functor F and the forgetful functor R is always a Quillen adjunction. If the counit $X \rightarrow RF(X)$ is a weak equivalence in \mathcal{C} for all cofibrant $X \in \mathcal{C}$, then the adjunction is a Quillen equivalence.

Proof. The first statement is a direct consequence of the description of generating (acyclic) cofibrations, the second follows immediately from the assumption and the 2-out-of-3 property. \square

Corollary 5.10 If the model structure on \mathcal{C} is right proper, then the model structure on $\mathcal{C}[T]$ is right proper as well.

Proof. The forgetful functor R preserves pullbacks. It preserves and reflects weak equivalences and fibrations. \square

The lifting theorem, applied to the standard Quillen model category \mathcal{U} of spaces (with weak equivalences the weak homotopy equivalences and fibrations the Serre fibrations), yields the non-equivariant model structure for \mathcal{L} -spaces from Theorem 2.44. In a second step, the combination of the theorem and the equivalence $\mathcal{M}^* \simeq \mathcal{M}_*$ then establishes a non-equivariant model structure for $*$ -modules. These results first appeared in [1, Sect. 4.6].

Alternative proof of 2.44. Requirement (R1) is proven in [13, A.1.9], requirement (R2) is immediate: The class of weak homotopy equivalences is stable under pushouts, transfinite composition, and retracts, and it contains J_{non} as a subset. The model structure is topological by Cor. A.39, left proper by Lemma A.38 and right proper by Cor. 5.10. \square

Theorem 5.11 (Non-equivariant model structure for \mathcal{M}_* ; [1], Thm. 4.16) The category of $*$ -modules admits a cofibrantly generated model structure $(\mathcal{M}_*)_{\text{non}}$ with weak equivalences the weak homotopy equivalences of underlying spaces. A map $f: X \rightarrow Y$ is a fibration if $F_{\boxtimes_{\mathcal{L}}}(*, f): F_{\boxtimes_{\mathcal{L}}}(*, X) \rightarrow F_{\boxtimes_{\mathcal{L}}}(*, Y)$ is a fibration in $(\mathcal{L}\mathcal{U})_{\text{non}}$, i.e. a Serre fibration of spaces. Sets of generating (acyclic) cofibrations are given by $I_{\text{non}} \boxtimes_{\mathcal{L}} *$ (respectively $J_{\text{non}} \boxtimes_{\mathcal{L}} *$). The model structure is proper, topological and symmetric monoidal with respect to the box product.

Although this is just an application of the Transport Theorem, we lay bare the main steps of the proof for reasons of transparency. Many of the technical details are just special cases of the general proofs, which we therefore refer to.

Proof. Apply the lifting theorem to the free algebra functor $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{L}\mathcal{U} \rightarrow \mathcal{M}^*$. The conditions (R1) and (R2) will follow from much stronger statements that appear in the

course of the proof of our main theorem, see Proposition 5.19 and Theorem 5.24. Thus, we obtain a model structure on \mathcal{M}^* with weak equivalences and fibrations detected by the inclusion $R: \mathcal{M}^* \rightarrow \mathcal{LU}$. The equivalence of categories $\mathcal{M}^* \simeq \mathcal{M}_*$ then yields a model structure on \mathcal{M}_* . We still have to explain why it matches our description: By Corollary 2.57, a morphism $f: X \rightarrow Y$ of \mathcal{L} -spaces is a weak homotopy equivalence if and only if $f \boxtimes_{\mathcal{L}} *$ is a weak homotopy equivalence if and only if $F_{\boxtimes_{\mathcal{L}}}(*, f)$ is a weak homotopy equivalence. The generating sets of (acyclic) cofibrations can be chosen as displayed above because the composition $F_{\boxtimes_{\mathcal{L}}}(*, -) \boxtimes_{\mathcal{L}} *: \mathcal{LU} \rightarrow \mathcal{M}_*$ is naturally equivalent to the functor $-\boxtimes_{\mathcal{L}} *$ by Proposition 2.37. Since all generating (acyclic) cofibrations for $(\mathcal{LU})_{\text{non}}$ are h -cofibrations, the model structure $(\mathcal{M}_*)_{\text{non}}$ is proper by Lemma 5.4. It is topological by 5.5 and symmetric monoidal by Proposition 5.6. \square

The space $\mathcal{L}(1)$ is (non-equivariantly) contractible, hence the natural map of spaces $A \rightarrow \mathcal{L}(1) \times A$ is a homotopy equivalence. By Corollary 5.9, all adjoint pairs in diagram 5.7 are Quillen equivalences with respect to the non-equivariant model structures for \mathcal{LU} and \mathcal{M}_* . In particular, the associated homotopy categories of \mathcal{L} -spaces and $*$ -modules are equivalent to the classical homotopy category $\text{Ho}\mathcal{U}$ of spaces.

Remark 5.12 The non-equivariant model structures for \mathcal{LU} and \mathcal{M}_* are just a small part of a big picture that relates several modern non-equivariant models for the stable homotopy category and their corresponding unstable analogues. Let \mathbb{IU} be the category of diagram spaces indexed on the category \mathbb{I} of finite sets and injective functions, and write $\Sigma\mathcal{S}, \mathcal{IS}, \mathcal{S}[\mathbb{I}], \mathcal{M}^{\mathcal{S}}$ and $\mathcal{M}_{\mathcal{S}}$, respectively, for the categories of symmetric spectra, orthogonal spectra, \mathbb{I} -spectra, co- \mathcal{S} -modules and \mathcal{S} -modules, respectively. Then these are related by adjoint pairs of functors as indicated in the following diagram.

$$\begin{array}{ccccccccc}
 \Sigma\mathcal{S} & \rightleftarrows & \mathcal{IS} & \rightleftarrows & \mathcal{S}[\mathbb{I}] & \rightleftarrows & \mathcal{M}^{\mathcal{S}} & \rightleftarrows & \mathcal{M}_{\mathcal{S}} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \mathbb{IU} & \rightleftarrows & \mathcal{IU} & \rightleftarrows & \mathcal{LU} & \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{R} \end{array} & \mathcal{M}^* & \begin{array}{c} \xleftarrow{-\boxtimes_{\mathcal{L}}*} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(*,-)} \end{array} & \mathcal{M}_*
 \end{array}$$

All of the involved categories admit non-equivariant model structures such that all horizontal adjunctions become Quillen equivalences. A very nice exposition is given by Lind in [8]. The vertical arrows are right Quillen functors. After passage to homotopy categories, these “infinite loop space” functors agree with the suspension-loop adjunction between spaces and Ω -spectra up to isomorphism of categories, cf. [8, Thm. 1.2].

The model structures on \mathbb{U} and \mathcal{U} are defined in terms of homotopy colimits, see [8, Thm. 1.1], so the latter can be seen as a non-equivariant version of the global model structure for orthogonal spaces, cf. the beginning of Section 3.3. The categories of \mathcal{L} -spaces and $*$ -modules carry the non-equivariant model structures described in this section.

All categories of spectra come with smash products, all categories of “structured spaces” with box products, such that these categories and the functors between them are (weakly) symmetric monoidal.

We refer the reader to [8] for a detailed analysis of all model categories and functors depicted in the diagram. Note that the functors between \mathcal{U} and $\mathcal{L}\mathcal{U}$ agree with our adjunction only “up to homotopy”; see [13, I.7.27] for a comparison of the different approaches.

5.3. Equivariant model categories of $*$ -modules

We have already seen that the Transport Theorem 5.3, applied to the non-equivariant model structure for \mathcal{L} -spaces, gives back the non-equivariant model structure for $*$ -modules. The theorem also applies to many of the equivariant model structure that we have seen in Section 2.6 and Chapter 4, and thus establishes a collection of interesting, Quillen equivalent model categories of $*$ -modules, the highlight being the sharp model structure. We spell out the details and examine the interactions of the different model structures.

The \mathcal{C} - and \mathfrak{F} -equivalences form admissible classes of weak equivalences in the sense of the Transport Theorem. All cofibrations and acyclic cofibrations are h -cofibrations, hence closed embeddings. The \mathcal{C} - and \mathfrak{F} -projective model structures for $\mathcal{L}\mathcal{U}$ are proper and topological. In addition, the latter is a weak symmetric monoidal model category. Thus, the Transport Theorem yields two model structures for $*$ -modules as follows.

Corollary 5.13 (\mathcal{C} -projective model structure for \mathcal{M}_*) The category of $*$ -modules admits a cofibrantly generated model structure $(\mathcal{M}_*)_{\mathcal{C}}$ with weak equivalences the global equivalences, and fibrations detected by the functor $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{M}_* \rightarrow (\mathcal{L}\mathcal{U})_{\mathcal{C}}$. Sets of generating cofibrations and acyclic cofibrations are given by $I_{\mathcal{C}} \boxtimes_{\mathcal{L}} *$ and $J_{\mathcal{C}} \boxtimes_{\mathcal{L}} *$. The model structure is proper and topological.

Corollary 5.14 (\mathfrak{F} -projective model structure for \mathcal{M}_*) The analogous statement is true with respect to the \mathfrak{F} -equivalences and \mathfrak{F} -fibrations. Furthermore, $(\mathcal{M}_*)_{\mathfrak{F}}$ is a symmetric monoidal model category.

where the vertical arrows form Quillen equivalences due to the Transport Theorem. As the weak equivalences in $(\mathcal{M}_*)_{\mathfrak{C}}$ and $(\mathcal{M}_*)_{\sharp}$ agree, we have established the following result.

Theorem 5.17 The different models for global homotopy theory are related by Quillen equivalences as follows:

$$\begin{array}{ccccccc}
 (\mathcal{U})_{\text{global}} & \xrightleftharpoons[u]{(-)(\mathbb{R}^\infty)} & (\mathcal{L}\mathcal{U})_{\sharp} & \xrightleftharpoons[\text{id}]{\text{id}} & (\mathcal{L}\mathcal{U})_{\mathfrak{C}} & \xrightleftharpoons[\Phi]{\Lambda} & O_{\text{gl}}\mathcal{U} \\
 & \searrow & \updownarrow \text{F}_{\boxtimes_{\mathcal{L}}}(*,-) & & \updownarrow \text{F}_{\boxtimes_{\mathcal{L}}}(*,-) & & \\
 & & (\mathcal{M}_*)_{\sharp} & \xrightleftharpoons[\text{id}]{\text{id}} & (\mathcal{M}_*)_{\mathfrak{C}} & &
 \end{array}$$

The square in the center commutes strictly, the triangle on the left commutes up to natural equivalence of functors. Moreover, the triangle consists of symmetric monoidal Quillen equivalences, i.e., a Quillen equivalences between (weak) symmetric monoidal model categories where the left adjoint is a strong monoidal functor.

Proof. The pair (Λ, Φ) is a Quillen equivalence by Proposition 3.36, the vertical pairs of arrows are Quillen equivalences by the Transport Theorem 5.3. It is easy to check on generating cofibrations and acyclic cofibrations that all other functor pairs are Quillen adjunctions; these are Quillen equivalences because the global equivalences of orthogonal spaces and \mathcal{L} -spaces are compatible by Lemma 3.27. The square in the center commutes for obvious reasons, the triangle on the left commutes up to natural equivalence of functors by Corollary 3.17. Up to natural equivalence, the diagonal adjunction is a composition of Quillen equivalences, and the left adjoint $(-)(\mathbb{R}^\infty)$ is strong monoidal by Proposition 3.16. The functor $-\boxtimes_{\mathcal{L}}*$ is strong monoidal by commutativity of $\boxtimes_{\mathcal{L}}$ and the isomorphism $*\boxtimes_{\mathcal{L}}* \cong *$. \square

Remark 5.18 Of course, there are even more nested inclusions of generating sets of cofibrations, e.g. $I_{\text{non}} \subseteq I_{\mathfrak{C}} \subseteq I_{\mathfrak{C}^b}$. These give rise to similar diagrams. All statements of Theorem 5.17 except for the last one remain true if the sharp model structures on $\mathcal{L}\mathcal{U}$ and \mathcal{M}_* are replaced by the flat ones. Moreover, there is a similar diagram if the \mathfrak{C} -projective model structures on $\mathcal{L}\mathcal{U}$ and \mathcal{M}_* and the model category $O_{\text{gl}}\mathcal{U}$ are replaced by their analogues with respect to all \mathfrak{F} -subgroups, but in this case the horizontal functor pairs in the middle are no Quillen equivalences anymore.

5.4. Proof of the main theorem

This section is dedicated to the proof of our main theorem, the Transport Theorem 5.3. It follows the same approach as in the non-equivariant case (see Section 5.2), but this time we start with the model structure $(\mathcal{L}\mathcal{U})_a$ as input. Given that the requirements hold, Theorem 5.8 yields an intermediate model structure $(\mathcal{M}^*)_a$ which then becomes the desired $(\mathcal{M}_*)_a$ under the equivalence of categories

$$\mathcal{M}^* \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}}^*} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(*, -)} \end{array} \mathcal{M}_*.$$

The weak equivalences in \mathcal{M}_* are those maps that are sent to maps in \mathcal{W} under the functor $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{M}_* \rightarrow \mathcal{L}\mathcal{U}$. Since an \mathcal{L} -map $f: X \rightarrow Y$ is in the class of weak equivalences \mathcal{W} if and only if $F_{\boxtimes_{\mathcal{L}}}(*, f)$ is, the class of weak equivalences in \mathcal{M}_* is precisely \mathcal{W}_* .

The counit of the adjunction

$$\mathcal{L}\mathcal{U} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{R} \end{array} \mathcal{M}^*.$$

we lift along is the map $\bar{\lambda}: X \rightarrow RF(X) = F_{\boxtimes_{\mathcal{L}}}(*, X)$ which is a strong \mathfrak{F} -equivalence by Corollary 2.56, hence is an element of \mathcal{W} . Corollary 5.9 implies that this adjunction is a Quillen equivalence, and the adjunction between $(\mathcal{M}^*)_a$ and $(\mathcal{M}_*)_a$ is a Quillen equivalence by our choice of the latter model structure.

We still have to verify the requirements (R1) and (R2) in order to guarantee that the lifting theorem applies. Recall that \mathbb{F} is the monad $RF: \mathcal{L}\mathcal{U} \rightarrow \mathcal{L}\mathcal{U}$ and I and J are sets of generating cofibrations and acyclic cofibrations for $(\mathcal{L}\mathcal{U})_a$, respectively. Then the requirements are:

- (R1) The domains of $I_{\mathbb{F}} = F(I)$ and $J_{\mathbb{F}} = F(J)$ are small relative to $I_{\mathbb{F}}\text{-cell}$ and $J_{\mathbb{F}}\text{-cell}$, respectively.
- (R2) Every morphism in $J_{\mathbb{F}}\text{-cell}$ is sent to a weak equivalence in $(\mathcal{L}\mathcal{U})_a$ under the forgetful functor.

The proof of (R2) relies only on the fact that the class of weak equivalences \mathcal{W} in $(\mathcal{L}\mathcal{U})_a$ is preserved by the mapping space functor $F = F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{L}\mathcal{U} \rightarrow \mathcal{M}^*$.

Proposition 5.19 Every cofibrantly generated model structure on $\mathcal{L}\mathcal{U}$ with an admissible class of weak equivalences \mathcal{W} satisfies requirement (R2).

Lemma 5.20 Let \mathcal{W}_0 be a class of morphisms in $\mathcal{L}\mathcal{U}$ which is stable under cobase change and is contained in \mathcal{W} . Let $i: A \rightarrow B$ be a morphism in \mathcal{W}_0 and

$$\begin{array}{ccc} F(A) & \longrightarrow & X \\ F(i) \downarrow & & \downarrow g \\ F(B) & \longrightarrow & Y \end{array}$$

a pushout square in \mathcal{M}^* . Then the underlying \mathcal{L} -map of g is in \mathcal{W} .

Proof. In order to show that the cobase change g of $F(i)$ is in \mathcal{W} , we will construct a “preimage” diagram in $\mathcal{L}\mathcal{U}$ that is sent to our given diagram under $F: \mathcal{L}\mathcal{U} \rightarrow \mathcal{M}^*$. Let $Y' \in \mathcal{L}\mathcal{U}$ be given as the following pushout:

$$\begin{array}{ccc} A & \longrightarrow & RX \\ i \downarrow & & \downarrow g' \\ B & \longrightarrow & Y' \end{array}$$

If we apply the left adjoint F , we obtain a pushout diagram

$$\begin{array}{ccc} F(A) & \longrightarrow & (FR)(X) \cong X \\ F(i) \downarrow & & \downarrow F(g') \\ F(B) & \longrightarrow & F(Y') \end{array}$$

in \mathcal{M}^* . Now $F(Y')$ and Y must be canonically isomorphic. Furthermore, under this isomorphism, $F(g')$ and g agree. The set \mathcal{W}_0 is stable under cobase change by assumption which implies $g' \in \mathcal{W}_0$. In particular, g' is in \mathcal{W} , hence so is $F(g') = g$. \square

Proof of Proposition 5.19. Let J be a set of generating acyclic cofibrations. In order to prove that the model structure satisfies (R2), we need to verify that each morphism in $J_{\mathbb{F}\text{-cell}}$ is sent to a morphism in \mathcal{W} under the forgetful functor, i.e., is a weak equivalence of underlying \mathcal{L} -spaces. As \mathcal{W} is stable under transfinite composition, it suffices to check that the underlying \mathcal{L} -maps of cobase changes (in \mathcal{M}^*) of morphisms in $J_{\mathbb{F}} = F(J)$ are contained in \mathcal{W} . This follows from Lemma 5.20 if we choose for \mathcal{W}_0 the class of acyclic cofibrations (which is stable under cobase changes in any model category, see e.g. [3, Prop. 3.14]). \square

Now we verify the smallness requirement (R1). We will show more generally that all counital \mathcal{L} -spaces are small with respect to sequences of closed embeddings. As all morphisms in the set of generating (acyclic) cofibrations of $(\mathcal{L}\mathcal{U})_a$ are closed embeddings by assumption, the same is true for transfinite compositions of cobase changes of these maps. Hence, requirement (R1) holds in this case.

Using the adjunction

$$(\mathcal{L}\mathcal{U})_a \begin{array}{c} \xrightarrow{F_{\boxtimes_{\mathcal{L}}}(*,-)} \\ \xleftarrow{R=\text{forget}} \end{array} (\mathcal{M}^*)_a$$

the smallness statement can be verified in the category of \mathcal{L} -spaces, which satisfies some smallness properties proven in Appendix A. If A is an \mathcal{L} -space which is κ -small relative to closed embeddings, then the functor $\mathcal{L}\mathcal{U}(A, -): \mathcal{L}\mathcal{U} \rightarrow \mathcal{U}$ commutes with λ -sequences along closed embeddings where λ is a κ -filtered ordinal. The same is true for the functor $\mathcal{L}\mathcal{U}(A, -): \mathcal{L}\mathcal{U} \rightarrow \mathcal{L}\mathcal{U}$ if A or the spaces in the sequence come with additional $\mathcal{L}(1)$ -actions. Each $A \in \mathcal{L}\mathcal{U}$ is at least $|A|$ -small, but it is even finite, if A is compact as a space, or if $A = \mathcal{L}/M$ for some closed submonoid $M \leq \mathcal{L}(1)$. For details, see Section A.2.

Corollary 5.21 Let κ be a cardinal and let $Y \in \mathcal{L}\mathcal{U}$ be κ -small relative to closed embeddings. The functor $F_{\boxtimes_{\mathcal{L}}}(Y, -): \mathcal{L}\mathcal{U} \rightarrow \mathcal{L}\mathcal{U}$ commutes with colimits along λ -sequences of closed embeddings of \mathcal{L} -spaces where λ is a κ -filtered ordinal. In particular, $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{L}\mathcal{U} \rightarrow \mathcal{L}\mathcal{U}$ commutes with colimits along all sequences of closed embeddings.

Proof. The \mathcal{L} -space $\mathcal{L}(2)$ is isomorphic to $\mathcal{L}(1)$, hence finite by Example A.24. Recall that we defined $F_{\boxtimes_{\mathcal{L}}}(Y, X) = \mathcal{L}\mathcal{U}(Y, \mathcal{L}\mathcal{U}(\mathcal{L}(2), X))$ and apply Corollary A.23 twice

(once for $A = \mathcal{L}(2)$ with the right action given from the second summand of $(\mathbb{R}^\infty)^2$, another time for $A = Y$ with the additional left action of $\mathcal{LU}(\mathcal{L}(2), X_\beta)$ given by the first summand of $(\mathbb{R}^\infty)^2$). \square

Corollary 5.22 Let λ be any limit ordinal. Colimits along λ -sequences X of closed embeddings of \mathcal{L} -spaces, where all X_β are counital, are again co- $*$ -modules.

Corollary 5.23 Let λ be any limit ordinal. Colimits along λ -sequences of closed embeddings in \mathcal{M}^* can be computed in \mathcal{LU} .

Theorem 5.24 Each $A \in \mathcal{M}^*$ is γ -small with respect to sequences of closed embeddings where γ is a cardinal such that the underlying \mathcal{L} -space of A is γ -small relative to closed embeddings of \mathcal{L} -spaces.

Proof. Let λ be a γ -filtered ordinal and M a λ -sequence of closed embeddings in \mathcal{M}^* . Write $F = F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{LU} \rightarrow \mathcal{M}^*$, let $R: \mathcal{M}^* \rightarrow \mathcal{LU}$ be the forgetful functor and note that $A \cong FR(A)$. Consider the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{colim}_{\beta} \mathcal{M}^*(FR(A), M_{\beta}) & \longrightarrow & \mathcal{M}^*(FR(A), \operatorname{colim}_{\beta} M_{\beta}) \\
\downarrow \cong & & \cong \downarrow \\
\operatorname{colim}_{\beta} \mathcal{LU}(R(A), R(M_{\beta})) & \longrightarrow & \mathcal{LU}(R(A), R(\operatorname{colim}_{\beta} M_{\beta})) \\
\operatorname{id} \downarrow = & & \cong \downarrow \diamond \\
\operatorname{colim}_{\beta} \mathcal{LU}(R(A), R(M_{\beta})) & \longrightarrow & \mathcal{LU}(R(A), \operatorname{colim}_{\beta} R(M_{\beta}))
\end{array}$$

The isomorphisms between the first and second row are given by the adjunction. The vertical isomorphism \diamond is given by Corollary 5.23. The horizontal map in the bottom row is a homeomorphism by Proposition A.22 and the smallness assumption, hence so is the map in the top row. \square

5.5. Categories of $\boxtimes_{\mathcal{L}}$ -monoids

In this section, we show how the sharp model structures for \mathcal{L} -spaces and for $*$ -modules and the global model structure for orthogonal spaces lift to the categories of monoids, modules and algebras defined in terms of $\boxtimes_{\mathcal{L}}$ and \boxtimes , respectively. The

monoidal Quillen equivalences between \mathcal{U} , \mathcal{LU} and \mathcal{M}_* then give rise to Quillen equivalences between the lifted model structures. The situation for commutative monoids is more difficult, as we explain in the end. For a definition of monoids, modules and algebras in symmetric monoidal categories, we refer the reader to [9, Ch. VII] and [14, Sect. 4].

We have seen in Chapter 2 that \mathcal{LU} and \mathcal{M}_* are closed (weak) symmetric monoidal categories with respect to $\boxtimes_{\mathcal{L}}$. Their associated categories of $\boxtimes_{\mathcal{L}}$ -monoids and commutative $\boxtimes_{\mathcal{L}}$ -monoids can alternatively be described as A_{∞} - and E_{∞} -spaces, respectively. Note that the linear isometries operad \mathcal{L} is an E_{∞} -operad with actions from the symmetric groups given by permutation of the n summands of $(\mathbb{R}^{\infty})^n$.

Proposition 5.25 ([1], Prop. 4.7) The category of A_{∞} -spaces structured by \mathcal{L} (considered as a nonsymmetric operad) is isomorphic to the category of $\boxtimes_{\mathcal{L}}$ -monoids in \mathcal{LU} . The category of E_{∞} -spaces structured by \mathcal{L} (considered as a symmetric operad) is isomorphic to the category of commutative $\boxtimes_{\mathcal{L}}$ -monoids in \mathcal{LU} .

Corollary 5.26 ([1], Sect. 4.4) The $\boxtimes_{\mathcal{L}}$ -monoids in \mathcal{M}_* are those A_{∞} -spaces which are $*$ -modules. The functor $-\boxtimes_{\mathcal{L}}*: \mathcal{LU} \rightarrow \mathcal{M}_*$ takes $\boxtimes_{\mathcal{L}}$ -monoids in \mathcal{LU} to $\boxtimes_{\mathcal{L}}$ -monoids in \mathcal{M}_* and the natural map $\lambda_X: X \boxtimes_{\mathcal{L}}* \rightarrow X$ is a map of $\boxtimes_{\mathcal{L}}$ -monoids if X is a $\boxtimes_{\mathcal{L}}$ -monoid. The analogous statement is true for commutative monoids and A_{∞} -spaces.

Remark 5.27 Blumberg, Cohen and Schlichtkrull write \boxtimes for the restriction of $\boxtimes_{\mathcal{L}}$ to the category of $*$ -modules, but we do not use this notation in order to avoid confusion with the box product of orthogonal spaces.

For a $\boxtimes_{\mathcal{L}}$ -monoid R , the category of left R -modules is isomorphic to the category $\mathcal{M}_*[T_R]$ of algebras over the monad $T_R: X \mapsto R \boxtimes_{\mathcal{L}} X$. If R is commutative, the same is true for the category of R -modules. Moreover, the category of $\boxtimes_{\mathcal{L}}$ -monoids is the category $\mathcal{M}_*[T]$ of algebras over the free monoid monad T given by

$$T: X \mapsto \coprod_{n \geq 0} X^{\boxtimes_{\mathcal{L}} n},$$

where the 0-th $\boxtimes_{\mathcal{L}}$ -power of X is the monoidal unit $*$. As R -algebras are monoids in the category of R -modules, these are algebras over the monad obtained by composing the free R -module functor, the free monoid functor and both forgetful functors.

We will use the theory developed in [14] to establish equivariant model structures for the categories of modules and algebras; for an account of non-equivariant model

structures, see [1, Thm. 4.18]. Let \mathcal{C} be a symmetric monoidal category under \otimes . Then [14, Thm. 4.1] gives sufficient conditions for a cofibrantly generated monoidal model structure on \mathcal{C} to lift to the associated categories of R -modules and R -algebras, respectively, where R is any (commutative) \otimes -monoid. One of the two conditions is the following monoid axiom.

Definition 5.28 Let \mathcal{C} be a cofibrantly generated monoidal model category. Let \mathcal{A} be the class of all maps $j \otimes Z$ where j is an acyclic cofibration and $Z \in \mathcal{C}$ is any object. We say that \mathcal{C} satisfies the *monoid axiom* if all relative \mathcal{A} -cell complexes are weak equivalences.

Lemma 5.29 The sharp model structures for \mathcal{L} -spaces and $*$ -modules satisfy the monoid axiom with respect to the box product.

Proof. Let $Z \in \mathcal{LU}$ be any object. Let $j: X \rightarrow Y$ be any acyclic cofibration in $(\mathcal{LU})_{\sharp}$. In particular, j is an h -cofibration and a global equivalence. This property is preserved by the functor $-\boxtimes_{\mathcal{L}} Z$, and it is stable under cobase changes and transfinite composition, so all relative \mathcal{A} -cell complexes are global equivalences. The same proof works for $(\mathcal{M}_*)_{\sharp}$. The Gluing Lemma A.38 is still true in \mathcal{M}_* because colimits are created in \mathcal{LU} according to Proposition 2.37. \square

The second hypothesis of [14, Thm. 4.1] is that all objects in the original category are small, but the authors explain in [14, Rem. 4.2] how this assumption can be weakened. In the case of the sharp model structure on \mathcal{M}_* , all possible sets of generating cofibrations I and generating acyclic cofibrations J consist of closed embeddings, hence so do their images I_T and J_T , respectively, under the free monoid functor. We need that the domains of maps in I_T (respectively J_T) are small relative to $I_T - cell$ (respectively $J_T - cell$). This can be checked in the underlying category of $*$ -modules, because the forgetful functor commutes with filtered direct limits. All statements remain true if the monad T is replaced by T_R . The desired smallness property now follows from the next lemma.

Lemma 5.30 All $*$ -modules are small relative to closed embeddings.

Proof. This is a formal consequence of the equivalence of categories $\mathcal{M}^* \simeq \mathcal{M}_*$, Corollary 5.21 and the fact that all co- $*$ -modules are small, see Theorem 5.24. \square

Now [14, Thm. 4.1] specializes to the following result.

Theorem 5.31 Consider the category of $*$ -modules equipped with the sharp model structure and let R be a $*$ -module.

- 1) If R is a $\boxtimes_{\mathcal{L}}$ -monoid, then the category of left R -modules is a cofibrantly generated model category.
- 2) If R is a commutative $\boxtimes_{\mathcal{L}}$ -monoid, then the category of R -modules is a cofibrantly generated monoidal model category satisfying the monoid axiom.
- 3) If R is a commutative $\boxtimes_{\mathcal{L}}$ -monoid, then the category of R -algebras is a cofibrantly generated model category.

In all cases, fibrations and weak equivalences are detected by the forgetful functor to $(\mathcal{M}_*)_{\sharp}$. Sets of generating cofibrations and acyclic cofibrations are given by the images of generating sets for \mathcal{M}_* under the free functor.

For $R = *$, the category of R -algebras is just the category of $\boxtimes_{\mathcal{L}}$ -monoids. It has a cofibrantly generated model structure by part 3) of the theorem.

Theorem 5.32 The analogue of Theorem 5.31 with respect to the monoidal model category $(\mathcal{LU})_{\sharp}$ is true.

Proof. A close inspection of the proof of [14, Thm. 4.1] shows that the first two statements do not require that the unital transformation is an isomorphism, so these hold because $(\mathcal{LU})_{\sharp}$ satisfies the monoid axiom. The proof of 3) makes use of the unital isomorphism in order to verify that all relative J_T -cell complexes are weak equivalences. We will give an alternative proof of this fact instead: All maps in J are h -cofibrations and global equivalences. The free monoid monad $T: X \mapsto \coprod_{n \geq 0} X^{\boxtimes_{\mathcal{L}} n}$ is continuous, hence preserves h -cofibrations. For a map $j: A \rightarrow B$ in J and $n \geq 2$, we can write the n -th summand $j^{\boxtimes_{\mathcal{L}} n}$ of $T(j)$ as the composition

$$(j \boxtimes_{\mathcal{L}} A^{\boxtimes_{\mathcal{L}} (n-1)}) \circ (B \boxtimes_{\mathcal{L}} j \boxtimes_{\mathcal{L}} A^{\boxtimes_{\mathcal{L}} (n-2)}) \circ \dots \circ (B^{\boxtimes_{\mathcal{L}} (n-1)} \boxtimes_{\mathcal{L}} j).$$

By Corollary 2.57, it is a global equivalence, hence so is the coproduct $T(j)$. The class of h -cofibrations which are global equivalences is closed under cobase change and transfinite composition, thus each morphism in $J_T - cell$ is a global equivalence. Smallness is not an issue because all \mathcal{L} -spaces are small relative to closed embeddings, and so relative to all images of cofibrations under T . \square

There are similar results for monoids, modules and algebras in the category of orthogonal spaces. The global model structure on $\mathcal{I}\mathcal{U}$ is symmetric monoidal, see [13, Prop. I.5.17], and it satisfies the monoid axiom, as shown in [13, Prop. I.8.24], where the \boxtimes -monoids are called *orthogonal monoid spaces*. It is easy to see that the data of an orthogonal monoid space is the same as that of a lax monoidal functor $\mathcal{I} \rightarrow \mathcal{U}$ with respect to orthogonal direct sum on \mathcal{I} and the categorical product on \mathcal{U} . Another application of [14, Thm. 4.1] yields the following statement.

Theorem 5.33 ([13], I.8.26) Let $R \in \mathcal{I}\mathcal{U}$ be an orthogonal monoid space.

- i) The category of left R -modules admits a model structure with fibrations and weak equivalences detected by the forgetful functor. If R is commutative, this model structure is monoidal and satisfies the monoid axiom.
- ii) Assume that R is commutative. Then the category of R -algebras admits a model structure with fibrations and weak equivalences detected by the forgetful functor. If the source of a cofibration of R -algebras is cofibrant as an R -module, then the map is a cofibration of R -modules. Every cofibrant R -algebra is also cofibrant as an R -module.

Note that all statements about model structures for left modules over a monoid hold for right modules, too. The only difference in the proof is that the free left R -module monad $X \mapsto R \otimes X$ has to be replaced by the free right module monad $X \mapsto X \otimes R$.

The left adjoint $(-)(\mathbb{R}^\infty)$ of the Quillen equivalence

$$(\mathcal{I}\mathcal{U})_{\text{global}} \begin{array}{c} \xrightarrow{(-)(\mathbb{R}^\infty)} \\ \xleftarrow{u} \end{array} (\mathcal{L}\mathcal{U})_{\#}$$

is a strong symmetric monoidal functor, so the adjunction is a *strong monoidal Quillen equivalence* in the sense of [15, Sect. 3], which is the follow-up paper of [14]. The authors explain in a more general context how the adjoint functors give rise to adjunctions between the associated categories of monoids and modules. In our context, the left and right adjoint restrict to the subcategory of monoids (respectively modules) where the monoid structure on $Y(\mathbb{R}^\infty)$ is defined in terms of the monoid structure on Y and the transformations which form the strong monoidal structure on the functor $(-)(\mathbb{R}^\infty)$. Instead of simply applying the main theorem of [15], we carry out the proof ourselves, because in our special case, it is rather simple. Before we state the comparison result, we check that the unit objects of $\mathcal{I}\mathcal{U}$ and $\mathcal{L}\mathcal{U}$ are cofibrant in the relevant model structures.

Lemma 5.34 The one-point orthogonal space $\mathbb{1}$ is cofibrant in the strong level model structure and in the global model structure on \mathcal{IU} .

Proof. The map $\mathbf{L}_{\{\text{id}\},\{0\}} \times i_0^Q$ is a generating cofibration for both of the model structures. Its value on $W \in \mathcal{I}$ is the map of spaces

$$\emptyset = \mathbf{L}(\{0\}, W) \times S^{-1} \rightarrow \mathbf{L}(\{0\}, W) \times D^0 \cong *,$$

hence the source is the constant empty orthogonal space, which is an initial object for \mathcal{IU} , whereas the target is $\mathbb{1}$. \square

Corollary 5.35 The singleton $*$ $\in \mathcal{LU}$ is cofibrant in the sharp model structures on the categories of \mathcal{L} -spaces and $*$ -modules.

Proof. The unique map $\emptyset \rightarrow *$ is the image of the cofibration $\mathbf{L}_{\{\text{id}\},\{0\}} \times i_0^Q$ under the left Quillen functor $(-)(\mathbb{R}^\infty)$. It is isomorphic to its own image under the left Quillen functor $-\boxtimes_{\mathcal{L}} *$. \square

Theorem 5.36 Consider the triangle of monoidal Quillen equivalences

$$\begin{array}{ccc} (\mathcal{IU})_{\text{global}} & \begin{array}{c} \xrightarrow{(-)(\mathbb{R}^\infty)} \\ \xleftarrow{u} \end{array} & (\mathcal{LU})_{\sharp} \\ & \begin{array}{c} \searrow \\ \swarrow \end{array} & \begin{array}{c} \uparrow \\ \downarrow \end{array} \\ & & (\mathcal{M}_*)_{\sharp} \end{array}$$

$(-)(\mathbb{R}^\infty)$ $F_{\boxtimes_{\mathcal{L}}}(*, -)$

between the global model structure for orthogonal spaces and the sharp model structures for \mathcal{L} -spaces and $*$ -modules. Let $B \in \mathcal{LU}$ be an orthogonal monoid space, let $M \in \mathcal{M}_*$ be a $\boxtimes_{\mathcal{L}}$ -monoid. Then the following holds with respect to the model structures from Theorem 5.31, 5.32 and 5.33:

- (1) The induced adjunctions between the categories of monoids in \mathcal{IU} , \mathcal{LU} and \mathcal{M}_* are Quillen equivalences.
- (2) If B is cofibrant as a monoid, then the induced adjunctions between the categories of right modules over B , $B(\mathbb{R}^\infty)$ and $B(\mathbb{R}^\infty) \boxtimes_{\mathcal{L}} *$ are Quillen equivalences.
- (3) The induced adjunctions between the categories of right modules over M , $F(M)$ and $(uF)(M)$ are Quillen equivalences.

Proof. In all cases, the forgetful functors preserve and reflect fibrations and weak equivalences, thus the lifted right adjoints are always right Quillen functors. The induced adjunctions over \mathcal{LU} and \mathcal{M}_* are Quillen equivalences because the functor $-\boxtimes_{\mathcal{L}}*$ preserves and reflects global equivalences and the counit $F(X)\boxtimes_{\mathcal{L}}* \rightarrow X$ is an isomorphism for all $X \in \mathcal{M}_*$. Up to natural isomorphism, the induced triangles commute, so it suffices to check that the horizontal adjunction induces Quillen equivalences of monoids and modules.

Assume that Y is cofibrant as an orthogonal monoid space, let $A \in \mathcal{LU}$ be a fibrant monoid. Then Y is cofibrant as an orthogonal space by Theorem 5.33 and A is fibrant as an \mathcal{L} -space by definition. So the categories of monoids are Quillen equivalent because the original categories were.

Let $Y \in \mathcal{IU}$ be a cofibrant B -module and let $X \in \mathcal{LU}$ be fibrant $B(\mathbb{R}^\infty)$ -module. Since the initial B -module B is cofibrant as an orthogonal space, so is Y by Theorem 5.33. By definition, X is fibrant in $(\mathcal{LU})_{\sharp}$. Now the Quillen equivalence condition holds in (2) and (3) because it does for the original categories. \square

Remark 5.37 Similar statements hold for the induced adjunctions between the categories of algebras over commutative monoids: With respect to the tensor product of modules, the model categories of modules are monoidal and the left Quillen functors between them are strong symmetric monoidal. Thus, a similar reasoning as in part (1) of the theorem yields Quillen equivalences between the categories of algebras.

The case of *commutative* monoids needs some extra work. The global model structure on \mathcal{IU} does not lift to a model structure for commutative monoids analogous to that from Theorem 5.33, see [13, Prop. II.4.3], but there is a *positive global model structure* with weak equivalences the global equivalences, and cofibrations those former cofibrations that induce a homeomorphism on the 0-th level. It is obviously Quillen equivalent to the global model structure via the identity functor pair. By [13, Thm. II.4.7], the positive global model structure satisfies the *commutative monoid axiom* which was introduced by White in [17, Def. 3.1]. In combination with the original monoid axiom and mild smallness requirements, this axiom is a sufficient condition for the existence of a model structure for commutative monoids detected by the forgetful functor in a way analogous to the (non-commutative) monoid case.

Theorem 5.38 ([13], Thm. II.4.9) The category of commutative \boxtimes -monoids in \mathcal{IU} (also called *ultra-commutative monoids*) admits a model structure with weak equivalences and fibrations detected by the forgetful functor to the positive global model category of orthogonal spaces. It is proper, topological and cofibrantly generated.

Unfortunately, we cannot use [15, Thm. 3.12] to compare the model category of ultra-commutative monoids to a (hypothetical) model category of commutative $\boxtimes_{\mathcal{L}}$ -monoids because the unit $\mathbb{1} \in \mathcal{U}$ is not cofibrant in the positive model structure: The map $\emptyset \rightarrow *$ is clearly not an isomorphism. One could analyze whether $(\mathcal{M}_*)_{\sharp}$ (or a similar model structure) satisfies the commutative monoid axiom, and whether it is Quillen equivalent to the model category of ultra-commutative monoids despite the failure of the cited theorem. We leave these questions open for further research.

A. Appendix: Background on monoid actions

This appendix provides some background material on spaces with monoid actions, including basic definitions and constructions and a collection of adjunctions. Moreover, we present a generalization of the well-known model structures for spaces with group actions to the monoid context. Much of the work that has been invested in the construction of model categories for \mathcal{L} -spaces and $*$ -modules (as described in [13] and in this Master's thesis) makes use of this tool.

A.1. Definitions and constructions

Throughout this appendix, let M be a topological monoid. Recall that we write \mathcal{U} for the category of (compactly generated weak Hausdorff) spaces.

Definition A.1 An M -space is a space $X \in \mathcal{U}$ which is equipped with a continuous left M -action. A morphism of M -spaces, or M -map, is an M -equivariant map of spaces. We write $M\mathcal{U}$ for the category of M -spaces and M -maps.

The category of M -spaces is a topological category, i.e., it is enriched over \mathcal{U} . The set $M\mathcal{U}(X, Y)$ is a closed subset of $\mathcal{U}(X, Y)$ and we give it the subspace topology.

Given two M -spaces X and Y , the product space $X \times Y$ has a canonical M -action via the diagonal action $m \cdot (x, y) = (m \cdot x, m \cdot y)$. The disjoint union $X \sqcup Y$ is an M -space in the obvious way. These are instances of the categorical product and sum, respectively. We will see a general description of arbitrary (co-)limits in Corollary A.13 in the next section.

In the following, we describe some constructions on M -spaces that appear ubiquitously in this thesis.

Definition A.2 Let X be a right M -space and let Y be a left M -space. The *balanced product* $X \times_M Y$ of X and Y is a choice of a coequalizer (in the category \mathcal{U} of spaces)

$$X \times M \times Y \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} X \times Y \longrightarrow X \times_M Y$$

where the upper and lower map are $(x, m, y) \mapsto (x \cdot m, y)$ and $(x, m, y) \mapsto (x, m \cdot y)$, respectively.

Notation A.3 Let $X \in M\mathcal{U}$, then we write X/M for a balanced product $* \times_M X$ with a singleton.

Remark A.4 Note that $X \times_M Y$ will often differ from the space $(X \times Y)/\sim$ where we quotient out the equivalence relation generated by $(x \cdot m, y) \sim (x, m \cdot y)$ because in general, a colimit taken in \mathcal{U} does not agree with the respective colimit taken in **Top**, not even as a set, cf. [16, Prop. 2.22]. The following lemma identifies a special case where the naïve description is still correct.

Lemma A.5 Let M be a topological monoid and $G \leq M$ a finite subgroup. Then $m \sim mg$ defines an equivalence relation on M and M/\sim is a model for the quotient space M/G .

Proof. The relation \sim can be viewed as a subset $E \subseteq M \times M$; it is obviously an equivalence relation. The second statement is true if the topological space M/\sim is a weak Hausdorff space. By [16, Cor. 2.21], this is equivalent to E being a closed subset of $M \times M$. But $M \times M = \bigcup_{g \in G} E_g$ where $E_g = \{(m, n) \in M \times M \mid n = mg\}$. The latter space is homeomorphic to the diagonal $\{(m, m)\} \subseteq M \times M$ via $(m, n) \mapsto (m, ng^{-1})$, so it is closed as a subset of $M \times M$. Now E is a finite union of closed subsets, hence closed. \square

Definition A.6 Let $\alpha: M \rightarrow N$ be a morphism of topological monoids and consider an M -space X and an N -space Y .

- 1) *Restriction* along α turns Y into an M -space $\alpha^*(Y) := Y$ with action $m \cdot y := \alpha(m) \cdot y$.
- 2) *Induction* along α turns X into an N -space $\alpha_*(X) := N \times_M X$ with action induced by the left action of N on itself. When forming the balanced product, we view N as a right M -space via $n \cdot m := n \cdot \alpha(m)$.

Definition A.7 Let $N \leq M$ be a submonoid, let $X \in M\mathcal{U}$. The N -fixed points of X are given by the space $X^N := \{x \in X \mid n \cdot x = x \forall n \in N\}$.

Definition A.8 Let $S \subseteq X$ be a subset of $X \in M\mathcal{U}$. The *stabilizer* of S is the submonoid

$$\text{stab}_M(S) := \{m \in M \mid m \cdot s = s \forall s \in S\}$$

The N -fixed points X^N form a closed subspace of X ; the stabilizer $\text{stab}_M(S)$ is a closed submonoid of M , see [13, Sect. A.1].

A.2. Formal properties

We collect some useful adjunctions involving M -spaces and the constructions from the last section.

Proposition A.9 The category $M\mathcal{U}$ of M -spaces is tensored and cotensored over the category \mathcal{U} of spaces, i.e., we have *tensors*

$$X \otimes A \in M\mathcal{U}$$

for all $X \in M\mathcal{U}$ and $A \in \mathcal{U}$ given by the product $X \times A$, where we view A as a trivial M -space, and *cotensors*

$$\Phi(A, Y) \in M\mathcal{U}$$

for all $A \in \mathcal{U}$ and $Y \in M\mathcal{U}$ given by $\mathcal{U}(A, Y)$ with M -action induced by the action on Y , which are subject to natural homeomorphisms

$$M\mathcal{U}(X \otimes A, Y) \cong \mathcal{U}(A, M\mathcal{U}(X, Y)) \cong M\mathcal{U}(X, \Phi(A, Y))$$

called the *tensor* and *cotensor adjunction*.

Remark A.10 Some authors require that certain natural transformations be isomorphisms, cf. [12, 16.3.6]. In our case, these conditions reduce to associativity of the product and the fact that \mathcal{U} is a closed symmetric monoidal category.

Instead of a proof, we provide a general recipe that can be used to prove most of the adjunctions in this section: Check that the natural homeomorphism $\mathcal{U}(A \times B, C) \cong \mathcal{U}(A, \mathcal{U}(B, C))$, which can be found e.g. in [16, Prop. 2.12], (co-)restricts to subspaces of equivariant maps as indicated in the desired adjunction in order to obtain well-defined mutual inverses. Continuity and naturality of these maps is immediate as (co-)restrictions of continuous maps are continuous.

Lemma A.11 For $Y \in M\mathcal{U}$, we have natural homeomorphisms

$$M\mathcal{U}(M, Y) \cong \mathcal{U}(*, Y) \cong Y,$$

where the first one is given by $\delta: f \mapsto (* \mapsto f(1))$, the second by evaluation on the unique element.

Proof. The natural homeomorphism on the right hand side is obvious. The action map $M \times Y \rightarrow Y$ is adjoint to a continuous map

$$Y \rightarrow \mathcal{U}(M, Y), \quad y \mapsto (m \mapsto m \cdot y).$$

Under the identification $\mathcal{U}(*, Y) \cong Y$, this map sends g to the unique M -map \hat{g} such that $\hat{g}(1) = g(*)$. The value of \hat{g} at $m \in M$ is dictated by the equivariance property:

$$\hat{g}(m) = \hat{g}(m \cdot 1) = m \cdot \hat{g}(1) = m \cdot g(*).$$

Thus, we have found a continuous inverse for δ . The map δ itself is the composition of the restriction map $M\mathcal{U}(M, Y) \rightarrow M\mathcal{U}(*, Y)$ and the inclusion $M\mathcal{U}(*, Y) \subseteq \mathcal{U}(*, Y)$, hence continuous. All constructions involved are natural. \square

Proposition A.12 The forgetful functor $M\mathcal{U} \rightarrow \mathcal{U}$ has a left adjoint given by the free M -space functor $M \times (-): \mathcal{U} \rightarrow M\mathcal{U}$, i.e.,

$$M\mathcal{U}(M \times A, Y) \cong \mathcal{U}(A, Y),$$

and a right adjoint given by the mapping space functor $\mathcal{U}(M, -): \mathcal{U} \rightarrow M\mathcal{U}$, i.e.,

$$\mathcal{U}(X, A) \cong M\mathcal{U}(X, \mathcal{U}(M, A)).$$

Both natural bijections are natural homeomorphisms.

Proof. For the first adjunction, combine the tensor adjunction and the isomorphism from Lemma A.11 to see that

$$M\mathcal{U}(M \times A, Y) \cong \mathcal{U}(A, M\mathcal{U}(M, Y)) \cong \mathcal{U}(A, Y).$$

For the second one, note that under the adjunction $\mathcal{U}(X \times M, A) \cong \mathcal{U}(X, \mathcal{U}(M, A))$, the subspace $M\mathcal{U}(X, \mathcal{U}(M, A))$ corresponds to a subspace $\mathcal{U}(X \times M, A)^+ \subseteq \mathcal{U}(X \times M, A)$ consisting of those maps f that satisfy $f(nx, m) = f(x, nm)$ for all elements $x \in X$ and $m, n \in M$. In particular, $f(x, m) = f(mx, 1)$ implies that f is determined by its restriction $f|_{X \times \{1\}}$. Similar reasoning as in the proof of the last lemma establishes a natural homeomorphism $\mathcal{U}(X, A) \cong \mathcal{U}(X \times M, A)^+$. \square

Corollary A.13 The category $M\mathcal{U}$ is bicomplete. Every (co-)limit of M -spaces is a (co-)limit in \mathcal{U} equipped with the (co-)limit M -action.

We present two more equivariant adaptations of $\mathcal{U}(A \times B, C) \cong \mathcal{U}(A, \mathcal{U}(B, C))$ dealing with the case of actions of two (not necessarily related) monoids.

Proposition A.14 Let M and N be topological monoids, X a right N -space, Y a left N -space, and $Z \in \mathcal{U}$. There is a natural homeomorphism

$$\mathcal{U}(X \times_N Y, Z) \cong N\mathcal{U}(Y, \mathcal{U}(X, Z))$$

whose value at f is the map that sends $y \in Y$ to the map $x \mapsto f([x, y])$. If in addition X and Z are left M -spaces, the (co-)restriction

$$M\mathcal{U}(X \times_N Y, Z) \cong N\mathcal{U}(Y, M\mathcal{U}(X, Z))$$

is a natural homeomorphism. The left N -action on $\mathcal{U}(X, Z)$ and $M\mathcal{U}(X, Z)$ is induced by the right action on X .

Proof. The proof follows our general recipe but must be handled with care since one of the spaces is given by a colimit construction. Consider the diagram

$$\begin{array}{ccccc} X \times N \times Y & \begin{array}{c} \xrightarrow{\mu_1} \\ \xrightarrow{\mu_2} \end{array} & X \times Y & \xrightarrow{p} & X \times_N Y \\ & & & \searrow & \downarrow \\ & & & & Z \end{array}$$

where p is the canonical map and $\mu_1(x, n, y) = (xn, y)$, $\mu_2(x, n, y) = (x, ny)$. By the universal property of the coequalizer, there is a natural homeomorphism

$$\mathcal{U}(X \times_N Y, Z) \cong M\mathcal{U}^{\text{coeq}}(X \times Y, Z), \quad f \mapsto ((x, y) \mapsto f([x, y]))$$

where the latter object is the space $\{f \mid f\mu_1 = f\mu_2\}$ of maps that coequalize the two parallel arrows. Under the homeomorphism

$$\mathcal{U}(X \times Y, Z) \cong \mathcal{U}(Y, \mathcal{U}(X, Z)),$$

it corresponds bijectively to the subset $\mathcal{U}^{\text{coeq}}(Y, \mathcal{U}(X, Z)) \subseteq \mathcal{U}(Y, \mathcal{U}(X, Z))$ of maps $g: Y \rightarrow \mathcal{U}(X, Z)$ that satisfy $g(ny)(x) = g(y)(xn)$. But these are precisely the N -equivariant maps, hence $\mathcal{U}^{\text{coeq}}(Y, \mathcal{U}(X, Z)) = N\mathcal{U}(Y, \mathcal{U}(X, Z))$. One readily verifies that restriction to subspaces of M -equivariant maps is well-defined, which implies the second statement. \square

Proposition A.15 Let M', M'' be topological monoids. Let X, Y and Z be left M' -, M'' -, and $(M' \times M'')$ -spaces, respectively. Then there is a natural homeomorphism

$$(M' \times M'')\mathcal{U}(X \times Y, Z) \cong M'\mathcal{U}(X, M''\mathcal{U}(Y, Z))$$

where $M''\mathcal{U}(Y, Z)$ is a left M' -space via the $(M' \times M'')$ -action on Z restricted along $M' \times \{1\} \rightarrow M' \times M''$.

Again, the proof follows our recipe.

Lemma A.16 Let $\alpha: N \rightarrow M$ be a continuous homomorphism of monoids and Z a left M -space. There is a natural homeomorphism of N -spaces

$$M\mathcal{U}(M, Z) \cong \alpha^*(Z), \quad f \mapsto f(1)$$

where the N -action on $M\mathcal{U}(M, Z)$ is induced by the right N -action on M via α .

Proof. The underlying space of $\alpha^*(Z)$ is just Z . Thus, the map is a homeomorphism by Lemma A.11. A simple calculation shows the N -equivariance. \square

We can now relate the restriction and induction functors.

Corollary A.17 Let $\alpha: N \rightarrow M$ be a map of topological monoids. Let $Y \in M\mathcal{U}$ and $A \in \mathcal{U}$ where we view the latter as a trivial N -space. There is a natural homeomorphism

$$M\mathcal{U}(M \times_N A, Y) \cong N\mathcal{U}(A, \alpha^*(Y)).$$

Proof. This is a combination of Proposition A.14 and Lemma A.16. \square

Corollary A.18 For each submonoid $N \leq M$, there are natural homeomorphisms

$$M\mathcal{U}(M/N, Y) \cong N\mathcal{U}(*, Y) \cong Y^N$$

called the “fixed points adjunction”.

Proof. The first identification is a special case of the previous corollary, the second is obvious. \square

Now we turn our attention to colimits along (transfinite) sequences of closed embeddings. We begin by recalling a standard result.

Lemma A.19 (cf. [7], Lemma 2.4.1, Prop. 2.4.2) Each topological space $A \in \mathbf{Top}$ is $|A|$ -small relative to the inclusions. Every compact space is finite relative to closed embeddings of (T1)-spaces.

In particular, every space A is $|A|$ -small relative to closed embeddings in the category \mathbf{Top} of all topological spaces. This remains true in \mathcal{U} as colimits along closed embeddings can be formed in the underlying category of topological spaces. We obtain:

Corollary A.20 Every M -space A is $|A|$ -small with respect to closed embeddings. It is finite if A is compact as a space.

Proof. Assume that λ is a $|A|$ -filtered ordinal, or any limit ordinal if A is a compact space. Let X be a λ -sequence of closed embeddings of M -spaces. We know that each M -map $f: A \rightarrow \operatorname{colim} X_\beta$ factors over $f_0: A \rightarrow X_\alpha$ for some $\alpha < \lambda$ in the underlying category \mathcal{U} . Since the canonical map $i: X_\alpha \rightarrow \operatorname{colim} X_\beta$ is injective and M -equivariant, it follows that f_0 is M -equivariant, too. \square

Lemma A.21 Let $K \in \mathcal{U}$ be a compact space. If $A \in \mathcal{LU}$ is finite relative to closed embeddings, then so is $A \times K$.

Proof. The functor $\mathcal{LU}(A \times K, -)$ is naturally isomorphic to the nested hom-space functor $\mathcal{U}(K, \mathcal{LU}(A, -))$ under the tensor adjunction. The statement of the lemma follows from the assumption on A and Lemma A.19. \square

Now we are able to prove the following “enriched” smallness statement.

Proposition A.22 Let κ be a cardinal and let $A \in M\mathcal{U}$ be any M -space which is κ -small relative to closed embeddings. The functor $M\mathcal{U}(A, -): M\mathcal{U} \rightarrow \mathcal{U}$ commutes with colimits along λ -sequences of closed embeddings of M -spaces where λ is a κ -filtered ordinal.

Proof. We have to prove that

$$j: \operatorname{colim}_{\beta < \lambda} M\mathcal{U}(A, X_\beta) \rightarrow M\mathcal{U}(A, \operatorname{colim}_{\beta < \lambda} X_\beta)$$

is a homeomorphism. Here $\operatorname{colim}_\beta X_\beta$ has the colimit M -action. Note that $M\mathcal{U}(A, -)$ preserves closed embeddings and consider the following diagram:

$$\begin{array}{ccc}
M\mathcal{U}(A, X_\beta) & \xhookrightarrow{j_\beta} & M\mathcal{U}(A, \operatorname{colim}_\beta X_\beta) \\
\downarrow i_\beta & \nearrow j & \\
\operatorname{colim}_\beta M\mathcal{U}(A, X_\beta) & &
\end{array}$$

By assumption, the map j is bijective. We may assume that $\operatorname{colim}_\beta M\mathcal{U}(A, X_\beta)$ is constructed in **Top**, hence its underlying set is a colimit in the category of sets. Continuity can be checked on the sets $M\mathcal{U}(A, X_\beta)$ by definition of the colimit topology, and each $j_\beta = j \circ i_\beta$ is a (continuous) closed embedding. It remains to show that j is a closed map: Let $S \subseteq \operatorname{colim}_\beta M\mathcal{U}(A, X_\beta)$ be a closed subset. Then each $i_\beta^{-1}(S) \subseteq M\mathcal{U}(A, X_\beta)$ and each $j_\beta(i_\beta^{-1}(S)) \subseteq M\mathcal{U}(A, \operatorname{colim}_\beta X_\beta)$ is closed. Consequently, $j(S) \subseteq M\mathcal{U}(A, \operatorname{colim}_\beta X_\beta)$ is closed. \square

The proposition extends to an equivariant statement:

Corollary A.23 Let κ be a cardinal and let A be an M -space which is κ -small relative to closed embeddings. Let X be a λ -sequence of closed embeddings of M -spaces where λ is an κ -filtered ordinal. Assume that in addition

- a) A has a right M -action, or
- b) all X_β have an additional left M -action such that the maps $X_\beta \rightarrow X_{\beta+1}$ are M^2 -equivariant.

Then the canonical homeomorphism from Proposition A.22 is M -equivariant, i.e., it is an isomorphism of M -spaces. The same is true for any filtered ordinal if A is a compact space.

Proof. Colimits in $M\mathcal{U}$ are constructed in \mathcal{U} , cf. Corollary A.13. In light of the previous proposition, it is enough to show that

$$j: \operatorname{colim}_{\beta < \lambda} M\mathcal{U}(A, X_\beta) \rightarrow M\mathcal{U}(A, \operatorname{colim}_{\beta < \lambda} X_\beta)$$

is M -equivariant. The maps $i_\beta: M\mathcal{U}(A, X_\beta) \rightarrow \operatorname{colim}_{\beta < \lambda} M\mathcal{U}(A, X_\beta)$ are equivariant as the colimit carries the colimit action. In case a), all $j_\beta: M\mathcal{U}(A, X_\beta) \rightarrow M\mathcal{U}(A, \operatorname{colim}_{\beta < \lambda} X_\beta)$ are equivariant because the functor $M\mathcal{U}(A, -): M\mathcal{U} \rightarrow M\mathcal{U}$ takes the maps $X_\beta \rightarrow$

$\operatorname{colim} X_\beta$ to M -maps; in case b), these maps themselves are M^2 -equivariant. The functor $M\mathcal{U}(A, -): (M^2)\mathcal{U} \rightarrow M\mathcal{U}$ preserves the additional M -actions, such that j_β is equivariant. In both cases, the equivariance of j follows. \square

Example A.24 Let $N \leq M$ be a closed submonoid, then the quotient space M/N is finite relative to closed embeddings of M -spaces: Let $X_0 \rightarrow X_1 \rightarrow \dots$ be such a sequence, then a map $f: M/N \rightarrow \operatorname{colim}_{\mathbb{N}} X_k$ is the same as an N -fixed point in $\operatorname{colim}_{\mathbb{N}} X_k$, hence a point in $\operatorname{colim}_{\mathbb{N}} X_k^N$ since we may assume that the colimit is constructed in **Top**, equipped with the colimit M -action. This point is represented by a point in some finite stage, hence f factors through some $f_0: M/N \rightarrow X_k$.

It is well-known that the fixed points functor, although being right adjoint, commutes with some well-behaved colimits. In the case of pushouts along closed embeddings, this is true for a larger class of functors by the following useful result.

Definition A.25 We say that a continuous functor $G: M\mathcal{U} \rightarrow \mathcal{U}$ is a *generalized fixed points functor* if the following holds:

- i) G takes topological embeddings to topological embeddings and closed embeddings to closed embeddings.
- ii) There is a natural closed embedding $\omega_X: GX \rightarrow X$.
- iii) G commutes with taking colimits along sequences of closed embeddings.
- iv) For each M -map $i: A \rightarrow B$ which is an embedding, the square

$$\begin{array}{ccc} GA & \xrightarrow{\omega} & A \\ Gi \downarrow & & \downarrow i \\ GB & \xrightarrow{\omega} & B \end{array}$$

has the *intersection property*, i.e., it satisfies $GA = A \cap GB$ if we replace all embeddings by actual inclusions of subsets.

Lemma A.26 For each closed submonoid $N \leq M$, the usual N -fixed points functor $(-)^N: \mathcal{L}\mathcal{U} \rightarrow \mathcal{U}$ is a generalized fixed points functor.

Proof. The functor $(-)^N \cong M\mathcal{U}(M/N, -)$ is continuous, preserves (closed) embeddings and comes with a natural closed inclusion $\omega_X: X^N \rightarrow X$. It commutes with sequences of closed embeddings by Proposition A.22 and Example A.24. If $h: W \rightarrow Z$ is an embedding, then

$$\begin{array}{ccc}
W^N & \xrightarrow{\omega} & W \\
h^N \downarrow & & \downarrow h \\
Z^N & \xrightarrow{\omega} & Z
\end{array}$$

has the intersection property because h is injective and equivariant, hence reflects subsets of fixed points. \square

Proposition A.27 Let $G: M\mathcal{U} \rightarrow \mathcal{U}$ be a generalized fixed points functor and consider a pushout diagram

$$\begin{array}{ccc}
A & \xrightarrow{f} & X \\
i \downarrow & & \downarrow j \\
B & \xrightarrow{g} & Y
\end{array}$$

in $M\mathcal{U}$. If i is a closed embedding, then the canonical map

$$\Phi: GX \cup_{GA} GB \rightarrow G(X \cup_A B)$$

is a homeomorphism.

Proof. The canonical map Φ is continuous. Since i is a closed embedding, so are j, Gi and Gj , and both pushouts can be taken in the category **Top** of all topological spaces, cf. [16, Prop. 2.35]. We may assume that all closed embeddings are actual inclusions of closed subspaces to keep the notation simple.

Let V be a closed subset of $GX \cup_{GA} GB$. Its preimage $(Gj)^{-1}(V)$ is closed in GX , hence closed in X (under the inclusion ω_X). Similarly, $(Gf)^{-1}(V)$ is closed in B . But this means that the image of V under Φ is closed in $X \cup_A B$. The latter space contains $G(X \cup_A B)$ as a closed subset, thus $\Phi(V)$ is closed in $G(X \cup_A B)$.

It remains to show bijectivity. Let y be in $G(X \cup_A B) \subseteq X \cup_A B$. As the pushouts are taken in **Top**, y must lie in the image of j or in the image of g . First assume that $y = j(x)$ for $x \in X$. The intersection property of

$$\begin{array}{ccc}
GX & \xrightarrow{\omega} & X \\
Gj \downarrow & & \downarrow j \\
G(X \cup_A B) & \xrightarrow{\omega} & X \cup_A B
\end{array}$$

implies $x \in GX \subseteq X$. Thus, the image of x in $GX \cup_{GA} GB$ is a preimage of y under Φ . Now let $y = g(b)$ for $b \in B - A$. The restriction $g|_{B-A} : B - A \rightarrow X \cup_A B$ is an embedding, hence the intersection property can be used again to construct a preimage. Finally, let p and p' be two points in $GX \cup_{GA} GB$ that map to the same point y in $G(X \cup_A B)$. We show that $p = p'$, proceeding case by case: If $p = (Gj)(x)$ and $p' = (Gj)(x')$ for $x, x' \in GX$, then p and p' agree, since Gj is injective. Let $p = (Gg)(b)$ and $p' = (Gg)(b')$ for $b, b' \in GX$. The images of p and p' agree in the pushout $G(X \cup_A B)$ taken in **Top**, hence either $b = b'$, in which case $p = p'$, or there must be $a, a' \in A$ such that $i(a) = b$, $i(a') = b'$ and $f(a) = f(a')$. The intersection property of

$$\begin{array}{ccc}
GA & \xrightarrow{\omega} & A \\
Gi \downarrow & & \downarrow i \\
GB & \xrightarrow{\omega} & B
\end{array}$$

implies that $a, a' \in GA \subseteq A$. Consequently, $p = G(gi)(a)$ and $p' = G(gi)(a')$ agree in the pushout $GX \cup_{GA} GB$. In the case $p = (Gj)(x)$, $p' = (Gg)(b')$ for $x \in GX$, $b' \in GB$, the proof is very similar. \square

A.3. Homotopy theory of M -spaces

Given any nice class \mathcal{C} of submonoids of M , there is a model structure on $M\mathcal{U}$ such that weak equivalences and fibrations are defined by testing on N -fixed points for all $N \in \mathcal{C}$. We need to specify the conditions on \mathcal{C} before we can state the precise result.

Definition A.28 (cf. [13], Def. A.1.2) A submonoid $N \leq M$ is *biclosed* if it is a closed subset of M and the following algebraic closure property is satisfied: For all $m \in M$ and $n \in N$, $mn \in N$ implies that $m \in N$.

Closed subgroups and stabilizer submonoids of M are always biclosed. Each submonoid $N \leq M$ has a biclosure \bar{N} obtained by intersecting all biclosed submonoids that contain N . For every $X \in MU$, all fixed-point spaces depend only on the biclosure of the according submonoid, i.e., we have $X^N = X^{\bar{N}}$ for all submonoids $N \leq M$, see [13, Sect. A.1].

Now let \mathcal{C} be a collection of biclosed submonoids. We will always assume that it is closed under conjugacy by invertible elements of M , since there is a homeomorphism of fixed point spaces

$$X^{N_1} \rightarrow X^{N_2}, x \rightarrow \phi \cdot x$$

whenever $N_2 = \phi N_1 \phi^{-1}$ are conjugate as submonoids of M .

Definition A.29 A morphism $f: X \rightarrow Y$ of M -spaces is a \mathcal{C} -equivalence (respectively \mathcal{C} -fibration) if the maps $f^N: X^N \rightarrow Y^N$ are weak homotopy equivalences (respectively Serre fibrations) for all $N \in \mathcal{C}$.

Theorem A.30 (cf. [13], Prop. A.1.10) The \mathcal{C} -equivalences and \mathcal{C} -fibrations form a model structure on the category of M -spaces, the \mathcal{C} -projective model structure $(MU)_{\mathcal{C}}$. It is proper, cofibrantly generated, and topological. Sets of generating (acyclic) cofibrations $I_{\mathcal{C}}$ (respectively $J_{\mathcal{C}}$) are given by

$$I_{\mathcal{C}} = \{M/N \times i_k^{\mathcal{O}} \mid N \in \mathcal{C}\}, \quad J_{\mathcal{C}} = \{M/N \times j_k^{\mathcal{O}} \mid N \in \mathcal{C}\}$$

We will need a slightly more general result in Section 4.3.

Definition A.31 Let \mathcal{O} be a family of M -spaces such that for each $W \in \mathcal{O}$, the functor $MU(W, -): MU \rightarrow \mathcal{U}$ is a generalized fixed points functor, see Definition A.25. We call an M -map $f: X \rightarrow Y$ an \mathcal{O} -equivalence (respectively \mathcal{O} -fibration) if the induced map $MU(W, f): MU(W, X) \rightarrow MU(W, Y)$ is a weak homotopy equivalence (respectively Serre fibration) for all $W \in \mathcal{O}$.

For $\mathcal{O} = \{M/N \mid N \in \mathcal{C}\}$, the \mathcal{O} -equivalences are precisely the \mathcal{C} -equivalences.

Lemma A.32 Let $X_0 \rightarrow X_1 \rightarrow \dots$ be a λ -sequence of M -spaces such that all maps are closed embeddings and \mathcal{O} -equivalences. Then the transfinite composition is an \mathcal{O} -equivalence.

Proof. Generalized fixed points functors commute with all sequential colimits along closed embeddings by definition. Weak homotopy equivalences are stable under transfinite composition. \square

Lemma A.33 Let J be a set of morphisms which are all \mathcal{L} -homotopy equivalences and h -cofibrations. Then all relative J -cell complexes are \mathcal{O} -equivalences.

Proof. The class of \mathcal{L} -homotopy equivalences is stable under forming coproducts. For $W \in \mathcal{O}$, the continuous functor $\mathcal{L}\mathcal{U}(W, -)$ sends \mathcal{L} -homotopy equivalences to homotopy equivalences of spaces, so all coproducts of maps in J are \mathcal{O} -equivalences. They are also h -cofibrations, hence all cobase changes of maps in J are \mathcal{O} -equivalences by the Gluing Lemma A.38, and the same is true for transfinite compositions of these maps by the previous lemma. \square

Theorem A.34 The \mathcal{O} -equivalences and \mathcal{O} -fibrations form a model structure on the category of M -spaces, the \mathcal{O} -projective model structure $(MU)_{\mathcal{O}}$. It is proper, cofibrantly generated, and topological. Sets of generating cofibrations $I_{\mathcal{O}}$ and acyclic cofibrations $J_{\mathcal{O}}$ are given by

$$I_{\mathcal{O}} = \{W \times i_k^{\mathcal{O}} \mid W \in \mathcal{O}\}, \quad J_{\mathcal{O}} = \{W \times j_k^{\mathcal{O}} \mid W \in \mathcal{O}\}$$

Proof. We mimick the proof of [13, Prop. A.1.10]. We define the class of cofibrations to be all maps that have the left lifting property with respect to the class of \mathcal{O} -acyclic fibrations, i.e., those \mathcal{O} -fibrations that are simultaneously \mathcal{O} -equivalences. Most model category axioms are immediate, only the factorization axiom and the other half of the lifting axiom remain.

We use the (transfinite) small object argument: Because of the adjunction

$$\mathcal{U} \begin{array}{c} \xrightarrow{M \times -} \\ \xleftarrow{\text{forget}} \end{array} MU,$$

the sets $I_{\mathcal{O}}$ and $J_{\mathcal{O}}$ detect \mathcal{O} -acyclic fibrations and \mathcal{O} -fibrations, respectively. All maps in $I_{\mathcal{O}}$ and $J_{\mathcal{O}}$ are closed embeddings, hence so are all maps of corresponding relative cell complexes. The functors $\mathcal{L}\mathcal{U}(W, -)$ commute with colimits along sequences of closed embeddings, so the objects $W \in \mathcal{O}$ are small, even finite, relative to closed embeddings. The same is then true for the domains of maps in $I_{\mathcal{O}}$ and $J_{\mathcal{O}}$ by Lemma A.21. Now the small object argument, applied to these sets, yields functorial factorizations as follows: Each \mathcal{L} -map factors as a relative $I_{\mathcal{O}}$ -cell complex followed by an \mathcal{O} -acyclic fibration, and as a relative $J_{\mathcal{O}}$ -cell complex followed by an \mathcal{O} -fibration. Clearly, all of these relative cell complexes are cofibrations, and relative $J_{\mathcal{O}}$ -cell complexes are also \mathcal{O} -equivalences by the preceding lemma.

For the other half of the lifting axiom, we have to show that if a map j is a cofibration and an \mathcal{O} -equivalence, then it has the left lifting property with respect to \mathcal{O} -acyclic fibrations. If we factor $j = qj'$ such that j' is a $J_{\mathcal{O}}$ -cell complex and q an \mathcal{O} -fibration, then the retract argument [6, Lemma 1.1.9] exhibits j as a retract of j' . Thus, it has the desired lifting property.

Right properness follows from right properness of \mathcal{U} and the fact that for each $W \in \mathcal{O}$, the generalized fixed points functor $M\mathcal{U}(W, -)$ preserves pullbacks as a right adjoint. Left properness is a consequence of Lemma A.38.

The model structure is topological because the pushout product of generating cofibrations $W \times i_k^{\mathcal{O}}$ for $M\mathcal{U}$ and $i_m^{\mathcal{O}}$ for \mathcal{U} is isomorphic to the generating cofibration $W \times i_{k+m}^{\mathcal{O}}$, and similar statements hold if one of the maps is replaced by a generating acyclic cofibration. \square

The most prominent special case is the standard model structure for G -spaces which is the projective model structure with respect to the class of all closed subgroups. We use it several times in order to define different notions of equivalences in the context of \mathcal{L} -spaces and orthogonal spaces.

Example A.35 (Standard model structure for GU) Let G be a topological group, then there is a model structure for GU such that $f: X \rightarrow Y$ is a weak equivalence (respectively fibration) if $f^H: X^H \rightarrow Y^H$ is a weak homotopy equivalence (respectively Serre fibration) for all closed subgroups $H \leq G$.

As $M\mathcal{U}$ is tensored over \mathcal{U} , see Proposition A.9, there is an associated notion of *M-homotopy*.

Definition A.36 By abuse of notation, write $I \in M\mathcal{U}$ for the unit interval under the trivial M -action. A map $H: I \times X \rightarrow Y$ of M -spaces is an *M-homotopy* between M -maps $f_0, f_1: X \rightarrow Y$ if $H(0, -) = f_0$ and $H(1, -) = f_1$.

As I is a trivial M -space, an M -homotopy is the same as an ordinary homotopy H between M -maps such that $H(t, -)$ is M -equivariant for each time $t \in I$. As usual, we define a map $f: X \rightarrow Y$ in $M\mathcal{U}$ to be an *M-homotopy equivalence* if it has an inverse up to M -homotopy.

Alternatively, we could define homotopies in terms of “cotensor path spaces” as maps $H: X \rightarrow \mathcal{U}(I, Y)$ such that $H(x)(0) = f(x)$ and $H(x)(1) = g(x)$. There is yet another way of defining (left and right) homotopies in terms of a given model structure on

$M\mathcal{U}$. The following lemma reconciles these concurring notions of homotopy in the case of the \mathcal{C} -projective model structures.

Lemma A.37 Let X and Y be M -spaces, then the two notions of homotopy between maps $X \rightarrow Y$ defined in terms of tensors and cotensors, respectively, agree. Assume that X is cofibrant in the \mathcal{C} -projective model structure, then the equivalence relations given by the following notions of homotopy between maps $X \rightarrow Y$ all coincide:

- a) Homotopy defined via “tensor cylinders” $X \times I$
- b) Homotopy defined via “cotensor path spaces” $\mathcal{U}(I, Y)$
- c) Left homotopy with respect to the \mathcal{C} -projective model structure
- d) Right homotopy with respect to the \mathcal{C} -projective model structure

Proof. The first statement is a consequence of the tensor and cotensor adjunctions, see Prop. A.9, and also shows that a) and b) coincide. Note that each $Y \in M\mathcal{U}$ is fibrant in the \mathcal{C} -projective model structure, hence c) and d) agree by general theory of model categories, see [3, Lemma 4.21]. The projection $I \times X \rightarrow X$ is a \mathcal{C} -equivalence since $(I \times X)^N = I \times X^N \rightarrow X^N$ is a homotopy equivalence of spaces. Hence, $I \times X$ is a cylinder object in the sense of [3, Def. 4.2]. This shows that a) and c) are identical because each choice of a cylinder object yields the same left homotopy relation. \square

Finally, we prove a generalized *Gluing Lemma* for $M\mathcal{U}$. Recall that an *h-cofibration* in a category tensored over \mathcal{U} is a morphism that has the *homotopy extension property*. More details and basic properties can be found in [13, Sect. I.5]. We only need to know that continuous functors preserve *h-cofibrations*.

Lemma A.38 (Gluing Lemma for $M\mathcal{U}$) Consider a pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow h \\ Z & \xrightarrow{k} & W \end{array}$$

in $M\mathcal{U}$ where f or g is an *h-cofibration*. If f is an \mathcal{O} -equivalence, see Definition A.31, or a homotopy equivalence of underlying spaces, then so is k .

Proof. We deal with the case of \mathcal{O} -equivalences first. Let W be an object in \mathcal{O} . An h -cofibration in $M\mathcal{U}$ is always a closed embedding, hence forming the pushout commutes with the generalized fixed points functor $M\mathcal{U}(W, -)$ in our case. We obtain a pushout

$$\begin{array}{ccc} M\mathcal{U}(W, X) & \xrightarrow{f_*} & M\mathcal{U}(W, Y) \\ g_* \downarrow & & \downarrow h_* \\ M\mathcal{U}(W, Z) & \xrightarrow{k_*} & M\mathcal{U}(W, W) \end{array}$$

in \mathcal{U} where f_* is a weak homotopy equivalence and one of the maps f_* or g_* is an h -cofibration of spaces. The gluing lemma for \mathcal{U} (see e.g. [12, Lemma 17.2.3] and note that the proof only uses the h -cofibration property) implies that k_* is a weak homotopy equivalence.

Now assume that f is a homotopy equivalence of underlying spaces, then so is k , because the Hurewicz model structure for \mathcal{U} is left proper, see e.g. [12, Thm. 17.1.1], and the forgetful functor $M\mathcal{U} \rightarrow \mathcal{U}$ preserves pushouts and h -cofibrations. \square

Corollary A.39 Assume that there is a model structure on $M\mathcal{U}$ with weak equivalences the \mathcal{C} -equivalences and cofibrations generated by a set

$$I = \{X_n \times i_k^{\mathcal{Q}}\}$$

where $(X_n)_n$ is a family of M -spaces. Then this model structure is topological.

Proof. As the pushout product $(X_n \times i_k^{\mathcal{Q}}) \square i_m^{\mathcal{Q}}$ is isomorphic to $X_n \times i_{k+m}^{\mathcal{Q}}$, the first part of the pushout product axiom holds. Now let $f: X \rightarrow Y$ be a cofibration in $M\mathcal{U}$ and $i: A \rightarrow B$ a cofibration in \mathcal{U} . Consider the commutative diagram

$$\begin{array}{ccccc} X \times A & \xrightarrow{X \times i} & & X \times B & \\ f \times A \downarrow & & \swarrow f' & & \downarrow f \times B \\ & & P & & \\ i' \nearrow & & \searrow q & & \\ Y \times A & \xrightarrow{Y \times i} & & Y \times B & \end{array}$$

where P is a pushout of $X \times i$ and $f \times A$. If f is a \mathcal{C} -equivalence, then so are $f \times A$ and $f \times B$. All cofibrations are h -cofibrations, so the Gluing Lemma implies that f' is a \mathcal{C} -equivalence, and the same is true for q by the 2-out-of-3 property. A similar reasoning applies if i is a weak homotopy equivalence, since we can regard i as a \mathcal{C} -equivalence between trivial M -spaces. \square

The last proof fails for \mathcal{O} -equivalences: In general, the tensor functor $X \times - : \mathcal{U} \rightarrow \mathcal{LU}$ might not take all weak homotopy equivalences to \mathcal{O} -equivalences.

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